

NON-ELEMENTARY K -QUASICONFORMAL GROUPS ARE LIE GROUPS

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ABSTRACT. Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and G is a non-elementary K -quasiconformal group. Then G is a Lie group acting on Ω .

Hilbert-Smith Conjecture states that every locally compact topological group acting effectively on a connected manifold must be a Lie group. Recently Martin [8] has solved the solution of the Hilbert-Smith Conjecture in the quasiconformal category (Theorem 1.2):

Theorem 1. *Let G be a locally compact group acting effectively by quasiconformal homeomorphisms on a Riemannian manifold. Then G is a Lie group.*

We will apply the Martin's theorem in this paper to show the following theorem.

Theorem 2. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and G is a non-elementary K -quasiconformal group. Then G is a Lie group acting on Ω .*

Let Ω and Ω' be domains in $\mathbb{R}^n$, $n \geq 2$. A homeomorphism $f: \Omega \rightarrow \Omega'$ is called to be K -quasiconformal if $f \in W_{loc}^{1,n}(\Omega, \mathbb{R}^n)$, the Sobolev space of functions whose first derivatives are locally L^n integrable, and for some $K < \infty$, f satisfies the differential inequality

$$(1) \quad |Df(x)|^n \leq KJ(x, f) \text{ almost everywhere in } \Omega.$$

Here $Df(x)$ is the derivative of f , $|Df(x)|$ is operator norm and $J(x, f)$ is the Jacobian determinant. We say f is quasiconformal if f is K -quasiconformal for some finite K . Thus, quasiconformal homeomorphisms are transformations which have uniformly bounded distortion. They provide a class of mappings

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that lie between homeomorphisms and conformal mappings. A quasiconformal homeomorphism of domain Ω in $\mathbb{R}^n$ can be extended to a subdomain in the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, for instance, by setting $f(\infty) = \infty$ [12].

Let Γ denote the family of all quasiconformal homeomorphisms of a domain Ω onto Ω' in $\overline{\mathbb{R}^n}$, then Γ forms a group under composition [1]. Let Γ_K denote the family of all K -quasiconformal homeomorphisms of a domain Ω onto Ω' in $\overline{\mathbb{R}^n}$. By contrast, Γ_K is not a group if $K > 1$. However, when $K = 1$, the family Γ_1 of all 1- quasiconformal self homeomorphisms of Ω in $\overline{\mathbb{R}^n}$ is the conformal group of Ω . Indeed, this group Γ_1 is a subgroup of the Möbius transformation group if $n > 2$ or if $n = 2$ with $\Omega = \overline{\mathbb{R}^n}$. In the latter case when $n = 2$ with $\Omega = \overline{\mathbb{R}^n}$, Γ_1 is just the classical Möbius transformation group, that is the group of linear fractional transformations of $\mathbb{C}$.

Let E be a non-empty subset of Ω , and define the *stabilizer* of a subset E :

$$(2) \quad \Gamma(E) = \{f \in \Gamma : f(E) = E\}$$

It is easy to see that $\Gamma(E)$ is a quasiconformal subgroup of Γ . And

$$(3) \quad \Gamma = \cup_{K \geq 1} \Gamma_K, \quad \Gamma(E) = \cup_{K \geq 1} \Gamma_K(E)$$

where $\Gamma_K(E) = \{f \in \Gamma_K : f(E) = E\}$.

A subfamily G of Γ_K is called a *K-quasiconformal group* if it constitutes a subgroup of Γ under composition. For example, the quasiconformal conjugate

$$G = f^{-1} \circ \Gamma_1 \circ f$$

of a subgroup of Möbius transformations Γ_1 of Ω' by a K -quasiconformal map $f: \Omega \rightarrow \Omega'$ is a K^2 -quasiconformal group acting on Ω . For subdomains of the plane Sullivan and Tukia showed in [9, 10], using a result of Maskit regarding groups of conformal transformations, that this is in fact the only construction. Namely a K -quasiconformal group of a domain $\Omega \subset \overline{\mathbb{R}^2}$ is quasiconformally conjugate to a subgroup of Möbius transformations of a domain $\Omega' \subset \overline{\mathbb{R}^2}$. The situation in higher dimensional is different, not every K -quasiconformal group is obtained in this way [7, 11].

As we know from Theorem 7.2 [3] that the compact-open topology of the space Γ of all quasiconformal homeomorphisms of a domain Ω onto Ω' in $\overline{\mathbb{R}^n}$ is equivalent to the topology induced from locally uniform convergence, where $\overline{\mathbb{R}^n}$ is a metric space with spherical metric. The space Γ is actually a metric space [4]. Therefore a compact subset coincides with a sequentially compact subset in Γ . And Γ possesses topological properties such as Hausdorff, normal and paracompact [3]. One of the most important aspects of quasiconformal homeomorphisms is their compactness properties. From now on every compact subset E of Ω in $\overline{\mathbb{R}^n}$ contains at least two points. We recall the following theorem from [4].

Theorem 3. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$, that G is a K -quasiconformal group of Ω acting on a compact subset E of Ω , and that $G \subset \Gamma_K(E)$. Then G is a locally compact topological transformation group.*

Notice that a manifold here is an n -dimensional smooth manifold (C^∞ differentiable) and it is also second countable, thus it is paracompact [13]. A smooth manifold is called a *Riemannian manifold* if there exists a Riemannian metric on it. However, on a paracompact smooth manifold there always exists a Riemannian metric [5], and a topological manifold is paracompact. Hence:

Proposition 1. *Every smooth manifold is a Riemannian manifold. In particular, every domain Ω in $\overline{\mathbb{R}^n}$ can be regarded as a Riemannian manifold.*

Suppose that G is a topological transformation group of a topological space X . For each $x \in X$, consider the subset of G :

$$(4) \quad G_x = \{g \in G : g(x) = x\}.$$

It is a subgroup of G which is called the *isotropy* subgroup of G at the point x of X . Similarly, consider the subset of G :

$$(5) \quad G_X = \{g \in G : g(x) = x, \text{ for all } x \in X\}.$$

It is a normal subgroup of G , and we have

$$(6) \quad G_X = \bigcap_{x \in X} G_x.$$

The topological transformation group G is said to act *effectively* on a topological space X if $G_X = \{e\}$. In the case that a topological transformation group G acts effectively on a topological space X , the corresponding group action is said to be *faithful* [2], i.e., the homomorphism

$$(7) \quad \phi: G \rightarrow \text{Homeo}(X), \text{ given by } g \mapsto g(x).$$

is faithful if ϕ is injective: $\text{Ker } \phi = \{e\}$. A topological transformation group may not act effectively on a topological space in general. But quasiconformal homeomorphisms are different, we have

Proposition 2. *Let G be a K -quasiconformal group of a domain Ω in $\overline{\mathbb{R}^n}$. Then G is a topological transformation group acting effectively on Ω .*

Proof. Notice that G is a topological transformation group [4]. Since $G \subset \text{Homeo}(\Omega)$, where $\text{Homeo}(\Omega)$ is the group of all homeomorphisms of Ω , consider the inclusion ϕ of G into $\text{Homeo}(\Omega)$, then ϕ is injective, i.e., $\text{Ker } \phi = \{e\}$. It is easy to see that $G_X = \text{Ker } \phi$. Thus $G_X = \{e\}$. $\square$

Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$, G is a K -quasiconformal group of Ω onto itself, and a compact subset E of Ω is invariant under G . Then the K -quasiconformal group G is a Lie group acting on Ω .

Theorem 4. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and $G \subset \Gamma_K(E)$ is a K -quasiconformal group. Then G is a Lie group acting on Ω .*

Proof. Apply Theorem 3, Proposition 1 and Proposition 2 to Theorem 1, we have the result. $\square$

A quasiconformal group G of self homeomorphisms of a domain Ω in $\overline{\mathbb{R}^n}$ is said to be *discontinuous* at a point $x \in \Omega$ if there exists a neighborhood U of x such that $g(U) \cap U = \emptyset$ for all but finite many $g \in G$. The *ordinary set* of G , denoted $O(G)$, is the set of all $x \in \Omega$ at which G is discontinuous. We say that G is a discontinuous group if $O(G) \neq \emptyset$. In other words, there exists one point of Ω which has a neighborhood that is carried outside of itself by all but finitely many elements of G . The complement of $O(G)$ is called the *limit set* of G and is denoted by $L(G)$: $L(G) = \Omega \setminus O(G)$. We say that G is an *elementary group* if the limit set $L(G)$ contains at most two points. Otherwise we say that G is *non-elementary*. Now it is ready for the main theorem mentioned at beginning.

Theorem 2. *Suppose that Ω is a subdomain of $\overline{\mathbb{R}^n}$ and G is a non-elementary K -quasiconformal group. Then G is a Lie group acting on Ω .*

Proof. Clearly, the ordinary set $O(G)$ is an open set in Ω hence in $\overline{\mathbb{R}^n}$. It follows that the limit set $L(G)$ is a closed set in Ω and $\overline{\mathbb{R}^n}$, thus $L(G)$ is a compact. Since the limit set $L(G)$ is invariant under G (Page 511, [6]), apply for $E = L(G)$ in Theorem 4, we immediately have the result. $\square$

This result leads to a natural question. Is the hypothesis of Theorem 2 that the group is non-elementary?

Indeed, Theorem 2 is held for an elementary K -quasiconformal group if its limit set $L(G)$ contains two points, because the subset E in Theorem 3 contains at least two points. Also, we believe that Theorem 2 will be true for an elementary K -quasiconformal group if its limit set $L(G)$ contains at most one point.

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