

G-CONTINUOUS FRAMES AND COORBIT SPACES

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ABSTRACT. A generalized continuous frame is a family of operators on a Hilbert space H which allows reproductions of arbitrary elements of H by continuous superpositions. Generalized continuous frames are natural generalization of continuous and discrete frames in Hilbert spaces which include many recent generalization of frames. In this article, we associate to a generalized continuous frame suitable Banach spaces, called generalized coorbit spaces, provided the frame satisfies a certain integrability condition. Also two classes of generalized coorbit spaces associated to a generalized continuous frame, its standard dual and some results are studied.

1. INTRODUCTION

Frames were first introduced in 1952 by Duffin and Schaeffer [5]. Frames have very important and interesting properties make them very useful in the characterization of function spaces, signal processing and many other fields. A discrete frame is a countable family of elements in a separable Hilbert spaces allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements [4]. Given a separable Hilbert spaces $\mathcal{H}$, a collections of elements $\{f_i\}_{i \in \mathbb{Z}}$ is called a discrete frame if there exist constants $0 < A_1, A_2 < \infty$ such that

$$A_1 \|f\|^2 \leq \sum_{i \in \mathbb{Z}} |\langle f, f_i \rangle|^2 \leq A_2 \|f\|^2 \text{ for all } f \in \mathcal{H}.$$

Later, this concept was generalized to continuous frames indexed by a Radon measure space [3, 2, 1] and [7]. For a locally compact Hausdorff space X endowed with a positive Radon measure μ , a family $\{\psi_x\}_{x \in X}$ of vectors in a separable Hilbert spaces $\mathcal{H}$ is called a continuous frame if there exist constants

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$0 < A_1, A_2 < \infty$ such that

$$A_1 \|f\|^2 \leq \int_X |\langle f, \psi_x \rangle|^2 d\mu(x) \leq A_2 \|f\|^2 \text{ for all } f \in \mathcal{H}.$$

The concept of generalized frames has been introduced by W. Sun [8]. Generalized frames are natural generalizations of frames as members of a Hilbert space to bounded linear operators. A family $\{\Lambda_i\}_{i \in \mathbb{Z}}$ of bounded linear operators from a separable Hilbert space $\mathcal{H}$ into another separable Hilbert space $\mathcal{K}$ is called a generalized frame if there are two positive constants A and B such that

$$A \|f\|^2 \leq \sum_{i \in \mathbb{Z}} \|\Lambda_i(f)\|^2 \leq B \|f\|^2 \text{ for all } f \in \mathcal{H}.$$

M. Fornasier and H. Rauhut have studied a kind of Banach spaces called coorbit spaces that vectors can be decomposed by use continuous frames [6]. Now we are going to extend this action by generalized continuous frames.

2. GENERALIZED CONTINUOUS FRAMES

Let X be a locally compact Hausdorff space endowed with a positive radon measure μ with $\text{supp } \mu = X$.

Definition 2.1. A family $\mathcal{F} = \{\Lambda_x\}_{x \in X}$ of bounded linear operator from a Hilbert space $\mathcal{H}$ into another Hilbert space $\mathcal{K}$ is called generalized continuous frame or simply g-continuous frame for $\mathcal{H}$ with respect to $\mathcal{K}$ if there are positive constants C_1 and C_2 such that

$$(1) \quad C_1 \|f\|^2 \leq \int_X \|\Lambda_x(f)\|^2 d\mu(x) \leq C_2 \|f\|^2 \text{ for all } f \in \mathcal{H}.$$

If $C_1 = C_2$ then the frame is called tight. We call $\mathcal{F}$ a g-continuous frame for $\mathcal{H}$ if $\mathcal{H} = \mathcal{K}$.

For the sake of simplicity we assume that the mapping $x \mapsto \Lambda_x$ is weakly continuous. Not that, if X is a countable set and μ is counting measure then we obtain the usual definition of (generalized discrete) frame. By the Riesz Representation Theorem, to every functional $\Lambda \in L(\mathcal{H}, \mathbb{C})$, one can find some $g \in \mathcal{H}$ such that $\Lambda(f) = \langle f, g \rangle$ for all $f \in \mathcal{H}$. Hence a continuous frame is equivalent to a g-continuous frame whenever $\mathcal{K} = \mathbb{C}$.

For a g-continuous frame $\mathcal{F}$ define the frame operator $S = S_{\mathcal{F}}$ in weak sense by

$$S: \mathcal{H} \rightarrow \mathcal{H}, \quad Sf := \int_X \Lambda_x^* \Lambda_x f d\mu(x)$$

where Λ_x^* is adjoint of the operator Λ_x .

Proposition 2.2. *The frame operator S is a bounded, positive, self-adjoint, and invertible.*

Proof. For all $f \in \mathcal{H}$

$$\begin{aligned}
 \langle Sf, f \rangle &= \int_X \langle \Lambda_x^* \Lambda_x f, f \rangle d\mu(x) = \int_X \langle \Lambda_x f, \Lambda_x f \rangle d\mu(x) \\
 &= \int_X \|\Lambda_x(f)\|^2 d\mu(x) \geq C_1 \|f\|^2 \geq 0 = \langle Sf, g \rangle \\
 &= \int_X \langle \Lambda_x^* \Lambda_x f, g \rangle d\mu(x) = \int_X \langle \Lambda_x f, \Lambda_x g \rangle d\mu(x) \\
 &= \int_X \langle f, \Lambda_x^* \Lambda_x g \rangle d\mu(x) = \langle f, Sg \rangle
 \end{aligned}$$

by (1) we have $C_1 \langle f, f \rangle \leq \langle Sf, f \rangle \leq C_2 \langle f, f \rangle$, $C_1 I \leq S \leq C_2 I$ and $\|I - C_2^{-1}S\| \leq 1 - \frac{C_1}{C_2} < 1$. Hence S is invertible operator. $\square$

Proposition 2.3. *Let $\mathcal{F} = \{\Lambda_x\}_{x \in X}$ is a g -continuous frame for Hilbert space $\mathcal{H}$ with frame operator S and bounds C_1, C_2 . Then $\tilde{\mathcal{F}} = \{\tilde{\Lambda}_x\}_{x \in X}$ such that $\tilde{\Lambda}_x = \Lambda_x S^{-1}$, is a frame for $\mathcal{H}$ with bounds C_1^{-1}, C_2^{-1} and frame operator S^{-1} .*

Proof. We show that $S^{-1}f = \int_X (S^{-1}\Lambda_x^* S^{-1}\Lambda_x)f d\mu(x)$.

$$\begin{aligned}
 \int_X (S^{-1}\Lambda_x^* S^{-1}\Lambda_x)f d\mu(x) &= S^{-1} \int_X (\Lambda_x^* S^{-1}\Lambda_x)f d\mu(x) \\
 &= S^{-1} \int_X \Lambda_x^* \Lambda_x (S^{-1}f) d\mu(x) \\
 &= S^{-1}(S(S^{-1}f)) = S^{-1}f
 \end{aligned}$$

also since $\mathcal{F} = \{\Lambda_x\}_{x \in X}$ is a frame for $\mathcal{H}$ then $C_1 I \leq S \leq C_2 I$. On other hand since I and S are self-adjoint and S^{-1} commutative with I and S ,

$$C_1 I S^{-1} \leq S S^{-1} \leq C_2 I S^{-1}$$

and hence

$$C_2^{-1} I \leq S^{-1} \leq C_1^{-1} I.$$

$\square$

If $\mathcal{F}$ is tight frame with bound $A = B = \lambda$ then $S = \lambda I$. Now the set

$$L^2(X, \mathcal{H}) := \{F: X \rightarrow \mathcal{H} \mid \int_X \|F(x)\|^2 d\mu(x) < \infty\},$$

with inner product $\langle F, G \rangle := \int_X \langle F(x), G(x) \rangle d\mu(x)$, is a Hilbert space.

We define the following two transformations associated to $\mathcal{F}$,

$$\begin{aligned}
 V: \mathcal{H} &\rightarrow L^2(X, \mathcal{H}), \quad Vf(x) := \Lambda_x(f), \\
 W: \mathcal{H} &\rightarrow L^2(X, \mathcal{H}), \quad Wf(x) := \Lambda_x(S^{-1}f).
 \end{aligned}$$

The operators V and W are well define since by (1) we have

$$\int_X \|Vf(x)\|^2 d\mu(x) = \int_X \|\Lambda_x(f)\|^2 d\mu(x) \leq C_2 \|f\|^2 < \infty$$

and

$$\begin{aligned} \int_X \|Wf(x)\|^2 d\mu(x) &= \int_X \|\Lambda_x(S^{-1}f)\|^2 d\mu(x) \\ &= \int_X \|S^{-1}\Lambda_x(f)\|^2 d\mu(x) \leq C_1^{-1} \|f\|^2 < \infty. \end{aligned}$$

In the following we show that adjoint operator of V and W given weakly by

$$V^*: L^2(X, \mathcal{H}) \rightarrow \mathcal{H}, \quad V^*F := \int_X \Lambda_y^* F(y) d\mu(y),$$

$$W^*: L^2(X, \mathcal{H}) \rightarrow \mathcal{H}, \quad W^*F := \int_X S^{-1} \Lambda_y^* F(y) d\mu(y).$$

Since for all $h \in \mathcal{H}$, we have

$$\begin{aligned} \langle V^*F, h \rangle &= \int_X \langle \Lambda_y^* F(y), h \rangle d\mu(y) = \int_X \langle F(y), \Lambda_y(h) \rangle d\mu(y) \\ &= \int_X \langle F(y), Vh(y) \rangle d\mu(y) = \langle F, Vh \rangle \end{aligned}$$

and

$$\begin{aligned} \langle W^*F, h \rangle &= \int_X \langle S^{-1} \Lambda_y^* F(y), h \rangle d\mu(y) \\ &= \int_X \langle F(y), \Lambda_y(S^{-1}h) \rangle d\mu(y) \\ &= \int_X \langle F(y), Wh(y) \rangle d\mu(y) = \langle F, Wh \rangle. \end{aligned}$$

Proposition 2.4. *Let $\mathcal{F} = \{\Lambda_x\}_{x \in X}$ is a g -continuous frame for Hilbert space $\mathcal{H}$ with frame operator S , then the following holds,*

- a) $S = V^*V$, $S^{-1} = W^*W$,
- b) $\langle f, g \rangle = \langle Vf, Wg \rangle = \langle Wf, Vg \rangle$,
- c) V and W are unitary if $\mathcal{F}$ is a tight frame,
- d) $\text{Range } V = \text{Range } W$,
- e) V and W are bijective transformations from $\mathcal{H}$ onto the Hilbert space $\mathcal{M}$ where

$$\mathcal{M} = \{F \in L^2(X, \mathcal{H}) : \int_X R(x, y) F(y) d\mu = F(x) \text{ a.e.}, R(x, y) = S^{-1} \Lambda_x \Lambda_y^*\}.$$

Proof. Let $f \in \mathcal{H}$, we have

$$(V^*V)(f) = \int_X \Lambda_y^* Vf(y) d\mu(y) = \int_X \Lambda_y^* \Lambda_y f d\mu(y) = Sf,$$

and

$$(V^*W)f = \int_X \Lambda_y^* W f(y) d\mu(y) = \int_X \Lambda_y^* \Lambda_y (S^{-1}f) d\mu(y) = S(S^{-1}f) = f.$$

In the same argument, $(W^*V)f = f$ and hence for all f and g in $\mathcal{H}$,

$$\langle f, g \rangle = \langle V f, W g \rangle.$$

Therefore a), b) and c) hold.

Since S is invertible and self-adjoint we have

$$f = SS^{-1}f = \int_X \Lambda_y^* \Lambda_y (S^{-1}f) d\mu(y) = \int_X \Lambda_y^* W f(y) d\mu(y),$$

and

$$f = S^{-1}Sf = \int_X S^{-1} \Lambda_y^* \Lambda_y f d\mu(y) = \int_X S^{-1} \Lambda_y^* V f(y) d\mu(y)$$

in the weak sense. Furthermore we have

$$\begin{aligned} Wf(x) &= \Lambda_x(S^{-1}f) = \Lambda_x S^{-1} \left(\int_X \Lambda_y^* W f(y) d\mu(y) \right) \\ &= \int_X \Lambda_x S^{-1} \Lambda_y^* W f(y) d\mu(y), \end{aligned}$$

$$Vf(x) = \Lambda_x(f) = \Lambda_x \left(\int_X S^{-1} \Lambda_y^* V f(y) d\mu(y) \right) = \int_X \Lambda_x S^{-1} \Lambda_y^* V f(y) d\mu(y).$$

Therefore

$$Wf(x) = \int_X R(x, y) W f(y) d\mu(y), \quad Vf(x) = \int_X R(x, y) V f(y) d\mu(y),$$

and hence Vf and Wf are in $\mathcal{M}$.

Conversely, let F be in $\mathcal{M}$ then

$$\begin{aligned} F(x) &= \int_X R(x, y) F(y) d\mu(y) = \int_X \Lambda_x S^{-1} \Lambda_y^* F(y) d\mu(y) \\ &= \Lambda_x S^{-1} \int_X \Lambda_y^* F(y) d\mu(y) = \Lambda_x S^{-1} (V^*F) = W(V^*F)(x). \end{aligned}$$

Therefore $F \in \text{Range } W$, $\mathcal{M} \subseteq \text{Range } W$ and $\mathcal{M} = \text{Range } W$. The same argument implies that $\mathcal{M} = \text{Range } V$. Finally by (1) V and W are injective and the proof is complete. $\square$

For every kernel function $K: X \times X \rightarrow L(\mathcal{H})$ and every function $F: X \rightarrow \mathcal{H}$ corresponds an operator K such that

$$(2) \quad K(F)(x) := \int_X K(x, y) F(y) d\mu(y).$$

Proposition 2.5. *Let $R: L^2(X, \mathcal{H}) \rightarrow L^2(X, \mathcal{H})$ and*

$$R(F)(x) := \int_X R(x, y)F(y)d\mu(y),$$

then,

- a) $R(x, y) = R(y, x)^*$ for all x and y in X ,
- b) $R(Vf) = Vf, R(Wf) = Wf$ for all f in $\mathcal{H}$,
- c) R is self-adjoint as an operator on $L^2(X, \mathcal{H})$,
- d) R is orthogonal projection from $L^2(X, \mathcal{H})$ onto $\mathcal{M}$.

Proof. a) and b) are trivial. R is self-adjoint as an operator on $L^2(X, \mathcal{H})$, since for all $F, G \in L^2(X, \mathcal{H})$ we have

$$\begin{aligned} \langle R(F), G \rangle &= \int_X \langle R(F)(x), G(x) \rangle d\mu(x) \\ &= \int_X \langle \int_X R(x, y)F(y)d\mu(y), G(x) \rangle d\mu(x) \\ &= \int_X \int_X \langle R(x, y)F(y), G(x) \rangle d\mu(y)d\mu(x) \\ &= \int_X \int_X \langle F(y), R(x, y)^*G(x) \rangle d\mu(y)d\mu(x) \\ &= \int_X \int_X \langle F(y), R(y, x)G(x) \rangle d\mu(y)d\mu(x) \\ &= \overline{\int_X \int_X \langle R(y, x)G(x), F(y) \rangle d\mu(y)d\mu(x)} \\ &= \overline{\int_X \int_X \langle R(y, x)G(x), F(y) \rangle d\mu(x)d\mu(y)} \\ &= \overline{\int_X \langle \int_X R(y, x)G(x)d\mu(x), F(y) \rangle d\mu(y)} \\ &= \overline{\int_X \langle R(G)(y), F(y) \rangle d\mu(y)} \\ &= \overline{\langle R(G), F \rangle} = \langle F, R(G) \rangle. \end{aligned}$$

For all $F \in L^2(X, \mathcal{H})$ we have $R(F) \in \text{Range}(R) = \text{Range}(V)$ then $R(F) = Vg$, for some $g \in \mathcal{H}$ and hence

$$R^2(F) = R(R(F)) = R(Vg) = Vg = R(F)$$

then $R^2 = R$ and

$$L^2(X, \mathcal{H}) = N(R) \oplus \text{Range}(R).$$

□

We assume in the following that $\|\Lambda_x\| \leq C$ for all $x \in X$. This implies $|Vf(x)| \leq C\|f\|$ and $|Wf| \leq C\|S^{-1}\|\|f\|$ for all $x \in X$ and, together with the weak continuity assumption, we conclude $Vf, Wf \in C^b(X, \mathcal{H})$ for all $f \in \mathcal{H}$, where $C^b(X, \mathcal{H})$ denotes the bonded continuous function of X to $\mathcal{H}$.

3. COORBIT SPACES

Associated to a g-continuous frames, there are Banach spaces called coorbit spaces where describe vectors in the Hilbert spaces of kernel functions. M. Fornasier and H. Rauhut [6] associated coorbit spaces to continuous frames. First, to build a weighted algebra, we need to introduce an special weight function.

Definition 3.1. Let m be a real weight function on $X \times X$. m is called admissible if,

- a) m is continuous,
- b) $1 \leq m(x, y) \leq m(x, z)m(z, y)$ for all $x, y, z \in X$,
- c) $m(x, y) = m(y, x)$ for all $x, y \in X$,
- d) $m(x, y) \leq C < \infty$ for all $x, y \in X$.

In order to get a weighted algebra we need to make a norm and a multiplication on kernel functions.

Proposition 3.2. Let

$$\mathcal{A}_1 := \{K : X \times X \rightarrow L(\mathcal{H}), K \text{ is measurable}, \|K|_{\mathcal{A}_1}\| < \infty\}$$

where

$$\|K|_{\mathcal{A}_1}\| := \max\{\text{ess sup}_{x \in X} \int_X \|K(x, y)\| d\mu(y), \text{ess sup}_{y \in X} \int_X \|K(x, y)\| d\mu(x)\}$$

is its norm (the norm in integral is uniform norm) and the multiplication in $\mathcal{A}_1$ is given by

$$K_1 \circ K_2(x, y) = \int_X K_1(x, z)K_2(z, y) d\mu(z),$$

such that in weak sense

$$K_1 \circ K_2(x, y) : \mathcal{H} \rightarrow \mathcal{H}, \quad K_1 \circ K_2(x, y)f = \int_X K_1(x, z)K_2(z, y)f d\mu(z).$$

Then $\mathcal{A}_1$ with $\|\cdot|_{\mathcal{A}_1}\|$ and the multiplication is a Banach algebra.

Proof. Obviously $\|\cdot|_{\mathcal{A}_1}\|$ is a norm and the conditions of an algebra satisfy. We prove the associativity of multiplication and completeness of the norm. For all $f \in \mathcal{H}$ and $K_1, K_2, K_3 \in \mathcal{A}_1$ we have

$$\begin{aligned} [K_1 \circ (K_2 \circ K_3)](x, y)f &= \int_X K_1(x, z)(K_2 \circ K_3)(z, y)f d\mu(z) \\ &= \int_X K_1(x, z) \int_X K_2(z, t)K_3(t, y)f d\mu(t) d\mu(z) \end{aligned}$$

$$\begin{aligned}
&= \int_X \int_X K_1(x, z) K_2(z, t) K_3(t, y) f d\mu(t) d\mu(z) \\
&= \int_X \int_X K_1(x, z) K_2(z, t) d\mu(z) K_3(t, y) f d\mu(t) \\
&= \int_X (K_1 \circ K_2)(x, t) K_3(t, y) f d\mu(t) \\
&= [(K_1 \circ K_2) \circ K_3](x, y) f,
\end{aligned}$$

and then $K_1 \circ (K_2 \circ K_3) = (K_1 \circ K_2) \circ K_3$.

Finally $\| \cdot \|_{\mathcal{A}_1}$ is Banach since if $\{K_n\}_{n=1}^\infty$ is Cauchy sequence in $\mathcal{A}_1$, then $\|K_n - K_m\|_{\mathcal{A}_1} \rightarrow 0$ as $m, n \rightarrow \infty$. Hence $\|K_n(x, y) - K_m(x, y)\|_{sup} \rightarrow 0$ as $m, n \rightarrow \infty$. Since $L(\mathcal{H})$ with uniform norm is Banach then there is $K \in L(\mathcal{H})$ such that $\|K_n(x, y) - K(x, y)\|_{sup} \rightarrow 0$ as $m, n \rightarrow \infty$ and hence $\|K_n - K\|_{\mathcal{A}_1} \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore $\mathcal{A}_1$ is a Banach algebra. $\square$

Now we define a corresponding weighted subalgebra respect to an admissible weight function m .

Proposition 3.3. *Let m be an admissible weight function and let,*

$$\mathcal{A}_m := \{K: X \times X \rightarrow L(\mathcal{H}), \quad Km \in \mathcal{A}_1\},$$

with the natural norm $\|K\|_{\mathcal{A}_m} := \|K_m\|_{\mathcal{A}_1}$. Then

- a) $\mathcal{A}_m$ is a Banach algebra,
- b) For every $K \in \mathcal{A}_m$, corresponding operator K on $L^2(X, \mathcal{H})$ defined by

$$K(F)(x) = \int_X K(x, y) F(y) d\mu(y),$$

is self adjoint.

Proof. Clearly, K is a linear operator on $L^2(X, \mathcal{H})$. For every $F, G \in L^2(X, \mathcal{H})$ we have

$$\begin{aligned}
\langle K(F), G \rangle &= \int_X \langle K(F)(x), G(x) \rangle d\mu(x) \\
&= \int_X \int_X \langle K(x, y) F(y), G(x) \rangle d\mu(y) d\mu(x) \\
&= \int_X \int_X \langle F(y), K^*(x, y) G(x) \rangle d\mu(y) d\mu(x) \\
&= \int_X \int_X \langle F(y), K(y, x) G(x) \rangle d\mu(y) d\mu(x) \\
&= \overline{\int_X \int_X \langle K(y, x) G(x), F(y) \rangle d\mu(y) d\mu(x)} \\
&= \overline{\int_X \langle \int_X K(y, x) G(x) d\mu(x), F(y) \rangle d\mu(y)}
\end{aligned}$$

$$\begin{aligned}
&= \overline{\int_X \langle K(G)(y), F(y) \rangle d\mu(y)} \\
&= \overline{\langle K(G), F \rangle} = \langle F, K(G) \rangle.
\end{aligned}$$

□

A function space Y that satisfies some properties is other tool for definition coorbit spaces associated to g-continuous frames.

Definition 3.4. Let $(Y, \|\cdot\|_Y)$ be a non-trivial Banach space of functions $F: X \rightarrow \mathcal{H}$ such that

- 1) Y is continuously embedded into $L^1_{loc}(X, \mathcal{H})$, where

$$L^1_{loc}(X, \mathcal{H}) := \{F: X \rightarrow \mathcal{H}, \int_K \|F(x)\| d\mu(x) < \infty$$

for every compact subset K of $X\}$,

- 2) If F is measurable and $G \in Y$ such that $\|F(x)\| \leq \|G(x)\|$ a.e. then $F \in Y$ and $\|F\|_Y \leq \|G\|_Y$.
 3) There exists an admissible weight function m such that $\mathcal{A}_m(Y) \subset Y$ and

$$\|K(F)\|_Y \leq \|K\|_{\mathcal{A}_m} \|F\|_Y$$

for all $K \in \mathcal{A}_m$, $F \in Y$, then $(Y, \|\cdot\|_Y)$ is called an m -function space.

In the rest of this article, let $(Y, \|\cdot\|_Y)$ with a weight function m be fixed. For fixed point $z \in X$ define a weight function on X by

$$\nu(x) := \nu_z(x) := m(x, z).$$

Now, we define the spaces

$$\begin{aligned}
\mathcal{H}^1_\nu &:= \mathcal{H}^1_\nu(X, \mathcal{H}) := \{f \in \mathcal{H}, Vf \in L^1_\nu(X, \mathcal{H})\}, \\
\mathcal{K}^1_\nu &:= \mathcal{K}^1_\nu(X, \mathcal{H}) := \{f \in \mathcal{H}, Wf \in L^1_\nu(X, \mathcal{H})\}
\end{aligned}$$

with natural norms

$$\|f\|_{\mathcal{H}^1_\nu} := \|Vf\|_{L^1_\nu}, \quad \|f\|_{\mathcal{K}^1_\nu} := \|Wf\|_{L^1_\nu}.$$

The frame operator S is an isometric isomorphism between $\mathcal{H}^1_\nu$ and $\mathcal{K}^1_\nu$.

Proposition 3.5. *The spaces $(\mathcal{H}^1_\nu, \|\cdot\|_{\mathcal{H}^1_\nu})$ and $(\mathcal{K}^1_\nu, \|\cdot\|_{\mathcal{K}^1_\nu})$ are Banach spaces.*

The proof is completely analogous to the proof of proposition 1 in [6] and hence omitted.

Now let R be in $\mathcal{A}_m$ and let $g \in \mathcal{H}$ then,

$$\begin{aligned}
\|\Lambda_y^* g\|_{\mathcal{K}^1_\nu} &= \int_X \|W(\Lambda_y^* g)(x)\| \nu(x) d\mu(x) \\
&= \int_X \|\Lambda_x S^{-1} \Lambda_y^* g\| \nu(x) d\mu(x) \leq \int_X \|R(x, y)\| \|g\| m(x, z) d\mu(x)
\end{aligned}$$

$$\leq \|g\| m(y, z) \int_X \|R(x, y)\| m(x, y) d\mu(x) \leq \|g\| \|R\| \mathcal{A}_m \| \nu(y),$$

and similarly

$$\begin{aligned} \|S^{-1}\Lambda_y^*g| \mathcal{H}_\nu^1\| &= \int_X \|V(S^{-1}\Lambda_y^*g)(x)\| \nu(x) d\mu(x) \\ &= \int_X \|\Lambda_x S^{-1}\Lambda_y^*g\| \nu(x) d\mu(x) \\ &\leq \|g\| m(y, z) \int_X \|R(x, y)\| m(x, y) d\mu(x) \leq \|g\| \|R\| \mathcal{A}_m \| \nu(y). \end{aligned}$$

Hence, $\Lambda_y^*g \in \mathcal{K}_\nu^1$ and $S^{-1}\Lambda_y^*g \in \mathcal{H}_\nu^1$ for all $y \in X$.

Now, we define the spaces

$$(\mathcal{H}_\nu^1)^\top := \{f: \mathcal{H}_\nu^1 \rightarrow \mathcal{H}, f \text{ is continuous and conjugat-linear}\},$$

$$(\mathcal{K}_\nu^1)^\top := \{f: \mathcal{K}_\nu^1 \rightarrow \mathcal{H}, f \text{ is continuous and conjugat-linear}\}.$$

Since $\Lambda_x^*g \in \mathcal{K}_\nu^1$ we may extend the transform V to $(\mathcal{K}_\nu^1)^\top$ by

$$Vf(x) = V_g f(x) = f(\Lambda_x^*g) = \langle f, \Lambda_x^*g \rangle, \quad f \in \mathcal{K}_\nu^1.$$

By the same argument, the transform W extends to $(\mathcal{H}_\nu^1)^\top$ by

$$Wf(x) = W_g f(x) = f(S^{-1}\Lambda_x^*g) = \langle f, S^{-1}\Lambda_x^*g \rangle, \quad f \in \mathcal{H}_\nu^1.$$

We may also extend the operator S to an isometric isomorphism between $(\mathcal{K}_\nu^1)^\top$ and $(\mathcal{H}_\nu^1)^\top$ by $\langle Sf, g \rangle = \langle f, Sg \rangle$ for $f \in (\mathcal{K}_\nu^1)^\top$ and $g \in \mathcal{H}_\nu^1$.

Definition 3.6. The coorbits of Y with respect the frame $\mathcal{F} = \{\Lambda_x\}_{x \in X}$ are defined as

$$\text{Co } Y := \text{Co}_g(\mathcal{F}, Y) := \{f \in (\mathcal{K}_\nu^1)^\top, \quad Vf = V_g f \in Y\},$$

$$\widetilde{\text{Co}} Y := \text{Co}_g(\tilde{\mathcal{F}}, Y) := \{f \in (\mathcal{H}_\nu^1)^\top, \quad Wf = W_g f \in Y\}$$

with natural norm

$$\|f| \text{Co } Y\| := \|Vf|Y\|, \quad \|f| \widetilde{\text{Co}} Y\| := \|Wf|Y\|.$$

The operator S is an isometric isomorphism between $\text{Co } Y$ and $\widetilde{\text{Co}} Y$.

There are some results in what follows and their proofs are similar to corresponding results in [6].

Proposition 3.7. Suppose that $R(Y) \subset L^\infty_{\frac{1}{\nu}}(X, \mathcal{H})$. Then the following statements hold.

- The spaces $(\text{Co } Y, \|\cdot\|_{\text{Co } Y})$ and $(\widetilde{\text{Co}} Y, \|\cdot\|_{\widetilde{\text{Co}} Y})$ are Banach spaces.
- A function $F \in Y$ is of the form Vf (resp. Wf) for some $f \in \text{Co } Y$ (resp. $\widetilde{\text{Co}} Y$) if and only if $F = R(F)$.

- c) The map $V: \text{Co}Y \rightarrow Y$ (resp. $W: \widetilde{\text{Co}}Y \rightarrow Y$) establishes an isometric isomorphism between $\text{Co}Y$ (resp. $\widetilde{\text{Co}}Y$) and the closed subspace $R(Y)$ of Y .

Corollary 1. *If Y also is a Hilbert space and $R(Y) \subset L_{\frac{1}{\nu}}^{\infty}(X, \mathcal{H})$ then $\text{Co}Y$ and $\widetilde{\text{Co}}Y$ are Hilbert spaces.*

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