

SOME PROPERTIES FOR INTEGRAL OPERATORS ON SOME ANALYTIC FUNCTIONS WITH COMPLEX ORDER

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ABSTRACT. In this paper we obtain some properties for two general integral operators on analytic functions with complex order.

1. INTRODUCTION

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane and denote by $\mathcal{H}(U)$, the class of the holomorphic functions in U . Consider A the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk U . In [7] Wiatrowski introduced the class of convex functions of order $b \in \mathbb{C}$ ($b \neq 0$) defined as follows:

Definition 1. A function $f(z) \in A$ is said to be convex function of order b , ($b \in \mathbb{C} - \{0\}$), that is, $f \in C(b)$, if and only if $f'(z) \neq 0$ in U and

$$(1) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z f''(z)}{f'(z)} \right\} > 0,$$

for all $z \in U$

In [6] Nasr and Aouf introduced the class $S(1-b)$, $b \neq 0$, complex, of starlike functions of order $1-b$, defined as follows:

Definition 2. A function $f(z) \in A$ is said to be starlike function of order $1-b$ ($b \in \mathbb{C} - \{0\}$), that is $f \in S(1-b)$, if and only if $f(z)/z \neq 0$ in U and

$$(2) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{z f'(z)}{f(z)} - 1 \right) \right\} > 0,$$

for all $z \in U$.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Integral operator, olomorphic functions, starlike functions, convex functions, complex order.

In [4] Frasin studied the classes $S_\alpha^*(b)$ and $C_\alpha(b)$, where the classes are defined as follows:

Definition 3. A function $f(z) \in \mathcal{A}$ is said to be a starlike of complex order b , ($b \in \mathbb{C} - \{0\}$) and type α , ($0 \leq \alpha < 1$), that is $f \in S_\alpha^*(b)$, if and only if

$$(3) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > \alpha,$$

for all $z \in U$.

Definition 4. A function $f(z) \in \mathcal{A}$ is said to be convex of complex order b , ($b \in \mathbb{C} - \{0\}$) and type α , ($0 \leq \alpha < 1$), that is $f \in C_\alpha(b)$, if and only if

$$(4) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \right\} > \alpha,$$

for all $z \in U$.

Remark 1. We note that for $\alpha = 0$ we have $C_0(b) = C(b)$ and $S_0^*(b) = S(1-b)$. Also we note that $S_\alpha^*(\cos \lambda e^{-i\lambda}) = S_\alpha^\lambda(\|\lambda\| < \frac{\pi}{2}, 0 \leq \alpha < 1)$, the class of λ -spirallike functions of order α , was introduced by Libera [5] and $C_\alpha(\cos \lambda e^{-i\lambda}) = C_\alpha^\lambda(\|\lambda\| < \frac{\pi}{2}, 0 \leq \alpha < 1)$ the class of λ -Robertson function of order α was introduced by Chichra [3].

In this paper, we consider two integral operators, the first operator is defined as follows:

$$(5) \quad F_n(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{f_n(t)}{t} \right)^{\alpha_n} dt \quad (\alpha_i > 0).$$

This operator was introduced by Breaz and Breaz [1] and this second operator is defined as follows:

$$(6) \quad F_{\alpha_1, \dots, \alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \dots (f_n'(t))^{\alpha_n} dt \quad (\alpha_i > 0).$$

This operator was introduced by Breaz, Owa and Breaz [2].

In this paper we shall study some properties for functions belonging to the classes $S_\alpha^*(b)$ and $C_\alpha(b)$.

2. MAIN RESULTS

Theorem 1. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$, and

$$(7) \quad 0 \leq 1 - \sum_{i=1}^n \alpha_i < 1.$$

We suppose that the functions $f_i \in S(1-b)$ for $i = \{1, \dots, n\}$ and $b \in \mathbb{C} - \{0\}$. Then we have the integral operator $F_n \in C_\gamma(b)$, where $\gamma = 1 - \sum_{i=1}^n \alpha_i$.

Proof. We calculate for F_n the derivatives of the first and second order. From (5) we obtain:

$$F'_n(z) = \left(\frac{f_1(z)}{z} \right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z} \right)^{\alpha_n}$$

and

$$F''_n(z) = \sum_{i=1}^n \alpha_i \left(\frac{zf'_i(z) - f_i(z)}{zf_i(z)} \right) F'_n(z).$$

Then we have

$$(8) \quad \begin{aligned} \frac{F''_n(z)}{F'_n(z)} &= \alpha_1 \left(\frac{zf'_1(z) - f_1(z)}{zf_1(z)} \right) + \cdots + \alpha_n \left(\frac{zf'_n(z) - f_n(z)}{zf_n(z)} \right), \\ \frac{F''_n(z)}{F'_n(z)} &= \alpha_1 \left(\frac{f'_1(z)}{f_1(z)} - \frac{1}{z} \right) + \cdots + \alpha_n \left(\frac{f'_n(z)}{f_n(z)} - \frac{1}{z} \right). \end{aligned}$$

Multiply the relation (8) with z we obtain:

$$(9) \quad \frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right).$$

Multiply the relation (9) with $\frac{1}{b}$ we obtain:

$$(10) \quad \frac{1}{b} \frac{zF''_n(z)}{F'_n(z)} = \frac{1}{b} \sum_{i=1}^n \alpha_i \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^n \alpha_i \left[1 + \frac{1}{b} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right] - \sum_{i=1}^n \alpha_i.$$

The relation (10) is equivalent to:

$$(11) \quad 1 + \frac{1}{b} \frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \alpha_i \left[1 + \frac{1}{b} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right] - \sum_{i=1}^n \alpha_i + 1.$$

By equalising the real parts of the above inequality we obtain:

$$(12) \quad \mathbf{Re} \left\{ 1 + \frac{1}{b} \frac{zF''_n(z)}{F'_n(z)} \right\} = \sum_{i=1}^n \alpha_i \mathbf{Re} \left[1 + \frac{1}{b} \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \right] - \sum_{i=1}^n \alpha_i + 1.$$

Since $f_i \in S(1-b)$ for $i = \{1, \dots, n\}$, we apply in the above relation the inequality (2) and obtain:

$$(13) \quad \mathbf{Re} \left\{ 1 + \frac{1}{b} \frac{zF''_n(z)}{F'_n(z)} \right\} > 1 - \sum_{i=1}^n \alpha_i.$$

Because $0 \leq 1 - \sum_{i=1}^n \alpha_i < 1$, we have that $F_n \in C_\gamma(b)$, where $\gamma = 1 - \sum_{i=1}^n \alpha_i$. $\square$

Corollary 1. *Let $\alpha_1 > 0$. If $0 \leq 1 - \alpha_1 < 1$ and the function $f_1 \in S(1-b)$, then the integral operator $F_1 \in \mathcal{C}_\rho(b)$, where $\rho = 1 - \alpha_1$.*

Proof. Putting $n = 1$ in Theorem 1 we obtain the result. $\square$

Putting $b = \cos \lambda e^{-i\lambda} (|\lambda| < \frac{\pi}{2})$ in Theorem 1, we obtain the following corollary:

Corollary 2. *Let the functions $f_i \in S^\lambda (|\lambda| < \frac{\pi}{2})$, for all $i \in \{1, 2, \dots, n\}$.*

Then the integral operator $F_n \in C^\lambda(\gamma)$, where $\gamma = 1 - \sum_{i=1}^n \alpha_i$, $\alpha_i > 0$ and

$$0 \leq 1 - \sum_{i=1}^n \alpha_i < 1.$$

Theorem 2. *Let the functions $f_i \in C(b)$, and $b \in C - \{0\}$, for all $i \in \{1, \dots, n\}$. Then the integral operator $F_{\alpha_1, \dots, \alpha_n} \in C_\eta(b)$, where $\eta = 1 - \sum_{i=1}^n \alpha_i$*

and $0 \leq 1 - \sum_{i=1}^n \alpha_i < 1$.

Proof. From (6), we have

$$(14) \quad \frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} = \alpha_1 \frac{f_1''(z)}{f_1'(z)} + \dots + \alpha_n \frac{f_n''(z)}{f_n'(z)}.$$

We multiply both sides of (14) with $\frac{z}{b}$, we obtain that

$$(15) \quad \frac{1}{b} \frac{z F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} = \alpha_1 \frac{1}{b} \frac{z f_1''(z)}{f_1'(z)} + \dots + \alpha_n \frac{1}{b} \frac{z f_n''(z)}{f_n'(z)}.$$

From the relation (15) we obtain that:

$$(16) \quad \operatorname{Re} \left(\frac{1}{b} \frac{z F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} + 1 \right) = \sum_{i=1}^n \alpha_i \operatorname{Re} \left(1 + \frac{1}{b} \frac{z f_i''(z)}{f_i'(z)} \right) - \sum_{i=1}^n \alpha_i + 1.$$

Since $f_i \in C(b)$, we have

$$(17) \quad \operatorname{Re} \left(\frac{1}{b} \frac{z F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} + 1 \right) > 1 - \sum_{i=1}^n \alpha_i.$$

Since $0 \leq 1 - \sum_{i=1}^n \alpha_i < 1$, the relation (17) implies that the integral operator

$F_{\alpha_1, \dots, \alpha_n} \in C_\eta(b)$, where $\eta = 1 - \sum_{i=1}^n \alpha_i$. □

Corollary 3. *Let the function $f_1 \in C(b)$. Then the integral operator $F_{\alpha_1} \in C_\sigma(b)$, where $\sigma = 1 - \alpha_1$ and $0 \leq 1 - \alpha_1 < 1$.*

Proof. Putting $n = 1$ in Theorem 2, we obtain the result. □

Remark 2. Putting $b = \cos \lambda e^{-i\lambda} (|\lambda| < \frac{\pi}{2})$ in Theorem 2, we have the following corollary:

Corollary 4. *Let the functions $f_i \in C^\lambda(|\lambda| < \frac{\pi}{2})$, for all $i \in \{1, 2, \dots, n\}$.*

Then the integral operator $F_{\alpha_1, \dots, \alpha_n} \in C^\lambda(\eta)$, where $\eta = 1 - \sum_{i=1}^n \alpha_i$ and $0 \leq 1 - \sum_{i=1}^n \alpha_i < 1$.

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Received July 27, 2007.

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