

LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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ABSTRACT. In this paper, we will study the continuity of multilinear commutator generated by Littlewood-Paley operator and Lipschitz functions on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

1. INTRODUCTION

Let T be the Calderón-Zygmund operator, Coifman, Rochberg and Weiss (see [4]) proves that the commutator $[b, T](f) = bT(f) - T(bf)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when T is replaced by the fractional operators. In [8, 16], Janson and Paluszynski study these results for the Triebel-Lizorkin spaces and the case $b \in \text{Lip}_\beta(R^n)$, where $\text{Lip}_\beta(R^n)$ is the homogeneous Lipschitz space. The main purpose of this paper is to discuss the boundedness of multilinear commutator generated by Littlewood-Paley operator and b_j on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where $b_j \in \text{Lip}_\beta(R^n)$.

2. PRELIMINARIES AND DEFINITIONS

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f , and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$, Q will denote a cube of R^n with side parallel to the axes. Let $f_Q = |Q|^{-1} \int_Q f(x) dx$ and $f^\#(x) = \sup_{y \in Q} |Q|^{-1} \int_Q |f(y) - f_Q| dy$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)$ ($0 < p \leq 1$) has the atomic decomposition characterization (see [12][17][18]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta, \infty}(R^n)$ be the homogeneous Triebel-Lizorkin space. The Lipschitz space $\text{Lip}_\beta(R^n)$ is the space

2000 *Mathematics Subject Classification.* 42B20, 42B25.

Key words and phrases. Littlewood-Paley operator; Multilinear commutator; Triebel-Lizorkin space; Herz-Hardy space; Herz space; Lipschitz space.

of functions f such that

$$\|f\|_{\text{Lip}_\beta} = \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Lemma 1 ([16]). *For $0 < \beta < 1$, $1 < p < \infty$, we have*

$$\begin{aligned} \|f\|_{\dot{F}_p^{\beta, \infty}} &\approx \left\| \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \\ &\approx \left\| \sup_{c \in \mathbb{R}} \inf_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - c| dx \right\|_{L^p}. \end{aligned}$$

Lemma 2 ([16]). *For $0 < \beta < 1$, $1 \leq p \leq \infty$, we have*

$$\begin{aligned} \|f\|_{\text{Lip}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q| dx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}. \end{aligned}$$

Lemma 3 ([2]). *For $1 \leq r < \infty$ and $\beta > 0$, let*

$$M_{\beta, r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}} \int_Q |f(y)|^r dy \right)^{1/r},$$

suppose that $r < p < n/\beta$, and $1/q = 1/p - \beta/n$, then

$$\|M_{\beta, r}(f)\|_{L^q} \leq C \|f\|_{L^p}.$$

Lemma 4 ([5]). *Let $Q_1 \subset Q_2$, then*

$$|f_{Q_1} - f_{Q_2}| \leq C \|f\|_{\text{Lip}_\beta} |Q_2|^{\beta/n}.$$

Definition 1. Let $0 < p \leq 1$. A function $a(x)$ on $\mathbb{R}^n$ is called a H^p -atom, if

- 1) $\text{supp } a \subset B(x_0, r)$ for some x_0 and for some $r > 0$ (or for some $r \geq 1$),
- 2) $\|a\|_{L^\infty} \leq |B(x_0, r)|^{-1/p}$,
- 3) $\int_{\mathbb{R}^n} a(x) x^\gamma dx = 0$ for all γ with $0 \leq |\gamma| \leq [n(1/p - 1)]$.

Lemma 5. (see [14]/[15]) *Let $0 < p \leq 1$. A distribution f on $\mathbb{R}^n$ is in $H^p(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the distributional sense, where each a_j is H^p -atom and λ_j are constants with $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Moreover,*

$$\|f\|_{H^p} = \inf \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p},$$

where the infimum take over all decompositions of f as above.

Definition 2. Let $0 < p, q < \infty$, $\alpha \in \mathbb{R}$, $B_k = \{x \in \mathbb{R}^n, |x| \leq 2^k\}$, $A_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

- 1) The homogeneous Herz space is defined

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L_{Loc}^q(R^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q}^p \right]^{1/p};$$

- 2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L_{Loc}^q(R^n) : \|f\|_{K_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q}^p + \|f \chi_{B_0}\|_{L^q}^p \right]^{1/p}.$$

Definition 3. Let $\alpha \in R$, $0 < p, q < \infty$.

- (1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}} = \|G(f)\|_{\dot{K}_q^{\alpha,p}};$$

- (2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$\|f\|_{HK_q^{\alpha,p}} = \|G(f)\|_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f , that is

$$G(f)(x) = \sup_{\varphi \in K_m} \sup_{|x-y| < t} |f * \varphi_t(y)|,$$

where $K_m = \{\varphi \in S(R^n) : \sup_{x \in R^n, |\alpha| \leq m} (1 + |u|)^{m+n} |D^\alpha \varphi(u)| \leq 1\}$, $\varphi_t(x) = t^{-n} \varphi(x/t)$ for $t > 0$, m is a positive integer and $S(R^n)$ is the Schwartz class (see [17, p. 88]).

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 4. Let $\alpha \in R$, $1 < q < \infty$. A function a on R^n is called a central (α, q) -atom (or a central (α, q) -atom of restrict type), if

- 1) $\text{supp } a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
- 2) $\|a\|_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
- 3) $\int_{R^n} a(x) x^\eta dx = 0$ for all η with $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Lemma 6 ([6, 15]). *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$ (or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q) -atoms (or central (α, q) -atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j , $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$\|f\|_{H\dot{K}_q^{\alpha,p}} \text{ (or } \|f\|_{HK_q^{\alpha,p}}) \approx \left(\sum_j |\lambda_j|^p \right)^{1/p}.$$

Definition 5 ([7]). Let $\alpha \in R$, $1 < p, q < \infty$.

- 1) A measure function f is said to belong to homogeneous weak Herz space $W\dot{K}_q^{\alpha,p}(R^n)$, if

$$\|f\|_{W\dot{K}_q^{\alpha,p}} = \sup_{\lambda > 0} \lambda \left(\sum_{k=-\infty}^{+\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q} \right)^{1/p} < \infty;$$

- 2) A measure function f is said to belong to inhomogeneous weak Herz space $WK_q^{\alpha,p}(R^n)$, if

$$\begin{aligned} \|f\|_{WK_q^{\alpha,p}} = \sup_{\lambda > 0} \lambda \left(\sum_{k=1}^{+\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q} \right. \\ \left. + |\{x \in B_0 : |f(x)| > \lambda\}|^{p/q} \right)^{1/p} < \infty. \end{aligned}$$

Definition 6. Let $\mu > 1$, $n > \delta > 0$ and ψ be a fixed function which satisfies the following properties:

- 1) $\int_{R^n} \psi(x) dx = 0$,
- 2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
- 3) $|\psi(x+y) - \psi(x)| \leq C|y|(1 + |x|)^{-(n+2-\delta)}$ when $2|y| < |x|$.

Given a positive integer m and the locally integrable function $b_j (1 \leq j \leq m)$. The multilinear commutator of Littlewood-Paley operator is defined by

$$g_{\mu,\delta}^{\vec{b}}(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x, y) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y - z) f(z) dz.$$

When $m = 1$, set

$$g_{\mu,\delta}^b(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t^b(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^b(f)(x, y) = \int_{R^n} (b(x) - b(z))\psi_t(y - z)f(z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define that

$$g_{\mu,\delta}(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

which is the Littlewood-Paley operator (see [18]).

Let H be the space $H = \left\{ h : \|h\| = \left(\int \int_{R_+^{n+1}} |h(y, t)|^2 dydt / t^{n+1} \right)^{1/2} < \infty \right\}$, then, for each fixed $x \in R^n$ $F_t(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H , and it is clear that

$$g_{\mu,\delta}(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t(f)(y) \right\|$$

and

$$g_{\mu,\delta}^{\vec{b}}(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^{\vec{b}}(f)(x, y) \right\|.$$

Note that when $b_1 = \dots = b_m$, $g_{\mu,\delta}^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 2, 3, 4, 8, 10, 9, 11, 16]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

Lemma 7 ([9]). *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $w \in A_1$. Then $g_{\mu,\delta}$ is bounded from $L^p(w)$ to $L^q(w)$.*

Given some functions b_j ($j = 1, \dots, m$) and a positive integer m and $1 \leq j \leq m$, we set $\|\vec{b}\|_{\text{Lip}_\beta} = \prod_{j=1}^m \|b_j\|_{\text{Lip}_\beta}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \dots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{\text{Lip}_\beta} = \|b_{\sigma(1)}\|_{\text{Lip}_\beta} \dots \|b_{\sigma(j)}\|_{\text{Lip}_\beta}$.

3. THEOREMS AND PROOFS

Theorem 1. *Let $0 < \beta < \min(1, 1/2m)$, $\mu > 3 + 1/n - 2\delta/n$, $1 < p < \infty$, $b_j \in \text{Lip}_\beta(R^n)$ for $1 \leq j \leq m$ and $g_{\mu,\delta}^{\vec{b}}$ be the multilinear commutator of Littlewood-Paley operator as in Definition 6. Then*

- (a) $g_{\mu,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $\dot{F}_p^{m\beta, \infty}(R^n)$ for $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$.
- (b) $g_{\mu,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1/p - 1/q = (m\beta + \delta)/n$ and $1/p > (m\beta + \delta)/n$.

Proof. (a). Fixed a cube $Q = (x_0, l)$ and $\tilde{x} \in Q$. Set $\vec{b}_Q = ((b_1)_Q, \dots, (b_m)_Q)$, where $(b_j)_Q = |Q|^{-1} \int_Q b_j(y) dy$, $1 \leq j \leq m$. Write $f = f\chi_Q + f_2 = f\chi_{R^n \setminus Q} = f_1 + f_2$, we have

$$\begin{aligned}
F_t^{\vec{b}}(f)(x, y) &= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_Q) - (b_j(z) - (b_j)_Q)] \psi_t(y - z) f(z) dz \\
&= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - \vec{b}_Q)_\sigma \int_{R^n} (\vec{b}(z) - \vec{b}_Q)_{\sigma^c} \psi_t(y - z) f(z) dz \\
&= (b_1(x) - (b_1)_Q) \dots (b_m(x) - (b_m)_Q) F_t(f)(y) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_Q) \dots (b_m - (b_m)_Q) f)(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - \vec{b}_Q)_\sigma \int_{R^n} (\vec{b}(z) - \vec{b}_Q)_{\sigma^c} \psi_t(y - z) f(z) dz \\
&= (b_1(x) - (b_1)_Q) \dots (b_m(x) - (b_m)_Q) F_t(f)(y) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_Q) \dots (b_m - (b_m)_Q) f_1)(y) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_Q) \dots (b_m - (b_m)_Q) f_2)(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} c_{m,j} (\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(y),
\end{aligned}$$

then

$$\begin{aligned}
&|g_{\mu,\delta}^{\vec{b}}(f)(x) - g_{\mu,\delta}(((b_1)_Q - b_1) \dots ((b_m)_Q - b_m)) f_2)(x_0)| \\
&\leq \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^{\vec{b}}(f)(x, y) \right. \\
&\quad \left. - \left(\frac{t}{t + |x_0 - y|} \right)^{n\mu/2} F_t(((b_1)_Q - b_1) \dots ((b_m)_Q - b_m) f_2)(y) \right\| \\
&\leq \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} (b_1(x) - (b_1)_Q) \dots (b_m(x) - (b_m)_Q) F_t(f)(y) \right\| \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} (\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(y) \right\| \\
&\quad + \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t((b_1 - (b_1)_Q) \dots (b_m - (b_m)_Q) f_1)(y) \right\|
\end{aligned}$$

$$\begin{aligned}
& + \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t \left(\prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right)(y) \right. \\
& \quad \left. - \left(\frac{t}{t + |x_0 - y|} \right)^{n\mu/2} F_t \left(\prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right)(y) \right\| \\
& = I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{1}{|Q|^{1+m\beta/n}} \int_Q |g_{\mu,\delta}^{\vec{b}}(f)(x) - g_{\mu,\delta}((b_1)_Q - b_1) \dots ((b_m)_Q - b_m) f_2)(x_0)| dx \\
& \leq \frac{1}{|Q|^{1+m\beta/n}} \int_Q I_1(x) dx + \frac{1}{|Q|^{1+m\beta/n}} \int_Q I_2(x) dx \\
& \quad + \frac{1}{|Q|^{1+m\beta/n}} \int_Q I_3(x) dx + \frac{1}{|Q|^{1+m\beta/n}} \int_Q I_4(x) dx \\
& = I + II + III + IV.
\end{aligned}$$

For I , by using Lemma 2, we have

$$\begin{aligned}
I & \leq \frac{1}{|Q|^{1+m\beta/n}} \sup_{x \in Q} |b_1(x) - (b_1)_Q| \dots |b_m(x) - (b_m)_Q| \int_Q |g_{\mu,\delta}(f)(x)| dx \\
& \leq C \|\vec{b}\|_{\text{Lip}_\beta} \frac{1}{|Q|^{1+m\beta/n}} |Q|^{m\beta/n} \int_Q |g_{\mu,\delta}(f)(x)| dx \\
& \leq C \|\vec{b}\|_{\text{Lip}_\beta} M(g_{\mu,\delta}(f))(\tilde{x}).
\end{aligned}$$

For II , taking $1 < r < p < q < n/\delta$, $1/q' + 1/q = 1$, $1/s' + 1/s = 1$, $1/q = 1/p - \delta/n$, $ps = r$ by using the Hölder's inequality and the boundedness of $g_{\mu,\delta}$ from L^p to L^q and Lemma 2, we get

$$\begin{aligned}
II & \leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma| |g_{\mu,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)| dx \\
& \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{q'} dx \right)^{1/q'} \times \\
& \quad \times \left(\frac{1}{|Q|} \int_{R^n} |g_{\mu,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f \chi_Q)(x)|^q dx \right)^{1/q} \\
& \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} \|\vec{b}_\sigma\|_{\text{Lip}_\beta} |Q|^{|\sigma|\beta/n} \frac{1}{|Q|^{1/q}} \times \\
& \quad \times \left(\int_{R^n} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c} f(x) \chi_Q(x)|^p dx \right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} \|\vec{b}_\sigma\|_{\text{Lip}_\beta} |Q|^{|\sigma|\beta/n} |Q|^{(-1/q)+(1/ps')+(1-\delta ps/n)/ps} \times \\
&\quad \times \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}|^{ps'} dx \right)^{1/ps'} \left(\frac{1}{|Q|^{1-\delta ps/n}} \int_Q |f(x)|^{ps} dx \right)^{1/ps} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} \|\vec{b}_\sigma\|_{\text{Lip}_\beta} |Q|^{|\sigma|\beta/n} \|\vec{b}_{\sigma^c}\|_{\text{Lip}_\beta} |Q|^{|\sigma^c|\beta/n} M_{\delta,r}(f)(\tilde{x}) \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For *III*, we choose $1 < r < p < q < n/\delta$, $1/q = 1/p - \delta/n$, $r = ps$, by the boundness of $g_{\mu,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with $1/s + 1/s' = 1$, we have

$$\begin{aligned}
III &\leq C \frac{1}{|Q|^{m\beta/n}} \left(\frac{1}{|Q|} \int_{R^n} |g_{\mu,\delta}(\prod_{j=1}^m (b_j - (b_j)_Q) f \chi_Q)(x)|^q dx \right)^{1/q} \\
&\leq C \frac{1}{|Q|^{m\beta/n}} \frac{1}{|Q|^{1/q}} \left(\int_{R^n} |\prod_{j=1}^m (b_j(x) - (b_j)_Q)|^p |f(x) \chi_Q(x)|^p dx \right)^{1/p} \\
&\leq C \frac{|Q|^{(-1/q)+1/ps'+(1-(\delta ps/n)/ps)}}{|Q|^{m\beta/n}} \left(\frac{1}{|Q|} \int_Q |\prod_{j=1}^m (b_j(x) - (b_j)_Q)|^{ps'} dx \right)^{1/ps'} \times \\
&\quad \times \left(\frac{1}{|Q|^{1-\delta ps/n}} \int_Q |f(x)|^{ps} dx \right)^{1/ps} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For *IV*, by the Minkowski's inequality and by the inequality $a^{1/2} - b^{1/2} \leq (a - b)^{1/2}$ for $a \geq b \geq 0$, we have

$$\begin{aligned}
&\left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t \left(\prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right)(y) \right. \\
&\quad \left. - \left(\frac{t}{t + |x_0 - y|} \right)^{n\mu/2} F_t \left(\prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right)(y) \right\| \\
&\leq \left[\int \int_{R_+^{n+1}} \left(\int_{(Q)^c} \left| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} - \left(\frac{t}{t + |x_0 - y|} \right)^{n\mu/2} \right| \right. \right. \\
&\quad \left. \left. \times \prod_{j=1}^m |b_j(z) - (b_j)_Q| |\psi_t(y - z)| |f(z)| dz \right)^2 \frac{dy dt}{t^{n+1}} \right]^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_{Q^c} \left[\int \int_{R_+^{n+1}} \left(\frac{t^{n\mu/2} |x - x_0|^{1/2} \prod_{j=1}^m |b_j(z) - (b_j)_Q| |\psi_t(y - z)| |f(z)|}{(t + |x - y|)^{(n\mu+1)/2}} \right)^2 \right. \\
&\quad \left. \times \frac{dy dt}{t^{n+1}} \right]^{1/2} dz \\
&\leq C \int_{Q^c} \prod_{j=1}^m |b_j(z) - (b_j)_Q| |f(z)| |x - x_0|^{1/2} \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \times \right. \\
&\quad \left. \times \frac{t^{-n} dy dt}{(t + |y - z|)^{2n+2-2\delta}} \right)^{1/2} dz.
\end{aligned}$$

Set $B = B(x, t)$, then

$$\begin{aligned}
&t^{-n} \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{dy}{(t + |y - z|)^{2n+2-2\delta}} \\
&\leq t^{-n} \left(\int_{B(x,t)} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{dy}{(t + |y - z|)^{2n+2-2\delta}} \right) \\
&\quad + t^{-n} \left(\sum_{k=1}^{\infty} \int_{2^k B \setminus 2^{k-1} B} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{dy}{(t + |y - z|)^{2n+2-2\delta}} \right) \\
&\leq C t^{-n} \left(\int_{B(x,t)} \frac{2^{2n+2-2\delta} dy}{(2t + |y - z|)^{2n+2-2\delta}} \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \int_{2^k B \setminus 2^{k-1} B} \left(\frac{t}{t + 2^{k-1} t} \right)^{n\mu+1} \frac{2^{(k+1)(2n+2-2\delta)} dy}{(2^{k+1} t + |y - z|)^{2n+2-2\delta}} \right) \\
&\leq C t^{-n} \left(\int_{B(x,t)} \frac{dy}{(t + |x - z|)^{2n+2-2\delta}} \right. \\
&\quad \left. + \sum_{k=1}^{\infty} 2^{(1-k)(n\mu+1)} \int_{2^k B} \frac{2^{k(2n+2-2\delta)} dy}{(t + |x - z|)^{2n+2-2\delta}} \right) \\
&\leq C t^{-n} \left(t^n + \sum_{k=1}^{\infty} 2^{-k(n\mu+1)} 2^{k(2n+2-2\delta)} (2^k t)^n \right) \frac{1}{(t + |x - z|)^{2n+2-2\delta}} \\
&\leq C \left(1 + \sum_{k=1}^{\infty} 2^{k(3n-n\mu+1-2\delta)} \right) \frac{1}{(t + |x - z|)^{2n+2-2\delta}} \\
&\leq \frac{C}{(t + |x - z|)^{2n+2-2\delta}}
\end{aligned}$$

and notice that $|x - z| \sim |x_0 - z|$ for $x \in Q$ and $z \in R^n \setminus Q$. We obtain

$$\begin{aligned}
& \int_{Q^c} \prod_{j=1}^m |b_j(z) - (b_j)_Q| |f(z)| |x - x_0|^{1/2} \times \\
& \quad \times \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{t^{-n} dy dt}{(t + |y - z|)^{2n+2-2\delta}} \right)^{1/2} dz \\
& \leq C \int_{Q^c} \prod_{j=1}^m |b_j(z) - (b_j)_Q| |f(z)| |x - x_0|^{1/2} \left(\int_0^\infty \frac{dt}{(t + |x - z|)^{2n+2-2\delta}} \right)^{1/2} dz \\
& \leq C \sum_{k=1}^\infty \int_{2^k Q \setminus 2^{k-1} Q} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} |f(z)| \prod_{j=1}^m |b_j(z) - (b_j)_Q| dz \\
& \leq C \sum_{k=1}^\infty 2^{-k/2} \left(\frac{1}{|2^k Q|^{1-\delta/n}} \int_{2^k Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_Q) \right|^{r'} dy \right)^{1/r'} \times \\
& \quad \times \left(\frac{1}{|2^k Q|^{1-\delta/n}} \int_{2^k Q} |f(y)|^r dy \right)^{1/r} \\
& \leq C \sum_{k=1}^\infty 2^{k(m\beta-1/2)} \|\vec{b}\|_{\text{Lip}_\beta} |Q|^{m\beta/n} M_{\delta,r}(f)(\tilde{x}) \\
& \leq C \|\vec{b}\|_{\text{Lip}_\beta} |Q|^{m\beta/n} M_{\delta,r}(f)(\tilde{x}),
\end{aligned}$$

so

$$IV \leq C \|\vec{b}\|_{\text{Lip}_\beta} M_{\delta,r}(f)(\tilde{x}).$$

We put these estimates together, by using Lemma 1 and taking the supremum over all Q such that $\tilde{x} \in Q$, we obtain

$$\begin{aligned}
\|g_{\mu,\delta}^{\vec{b}}(f)\|_{\dot{F}_q^{m\beta,\infty}} & \leq C \|\vec{b}\|_{\text{Lip}_\beta} (\|M(g_{\mu,\delta}(f)) + M_{\delta,r}(f)\|_{L^q}) \\
& \leq C \|\vec{b}\|_{\text{Lip}_\beta} (\|M(g_{\mu,\delta}(f))\|_{L^q} + \|M_{\delta,r}(f)\|_{L^q}) \\
& \leq C \|\vec{b}\|_{\text{Lip}_\beta} (\|g_{\mu,\delta}(f)\|_{L^q} + \|M_{\delta,r}(f)\|_{L^q}) \\
& \leq C \|\vec{b}\|_{\text{Lip}_\beta} \|f\|_{L^p}.
\end{aligned}$$

This completes the proof of (a).

(b). By some argument as in proof of (a), we have

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q |g_{\mu,\delta}^{\vec{b}}(f)(x) - g_{\mu,\delta}(((b_1)_Q - b_1) \dots ((b_m)_Q - b_m)f_2)(x_0)| dx \\
& \leq \frac{1}{|Q|} \int_Q I_1(x) dx + \frac{1}{|Q|} \int_Q I_2(x) dx + \frac{1}{|Q|} \int_Q I_3(x) dx + \frac{1}{|Q|} \int_Q I_4(x) dx \\
& = V_1 + V_2 + V_3 + V_4.
\end{aligned}$$

For V_1 , taking $1/s = 1/r - \delta/n$, by the boundedness of $g_{\mu,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$, so we have

$$\begin{aligned}
V_1 &\leq \frac{1}{|Q|} \sup_{x \in Q} |b_1(x) - (b_1)_Q| \dots |b_m(x) - (b_m)_Q| \int_Q |g_{\mu,\delta}(f)(x)| dx \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} |Q|^{m\beta/n} \left(\frac{1}{|Q|} \int_Q |g_{\mu,\delta}(f)(x)|^s dx \right)^{1/s} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} |Q|^{m\beta/n-1/s} \left(\int_Q |f(x)|^r dx \right)^{1/r} \\
&= C \|\vec{b}\|_{\text{Lip}_\beta} \left(\frac{1}{|Q|^{1-(m\beta+\delta)r/n}} \int_Q |f(x)|^r dx \right)^{1/r} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} M_{m\beta+\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For V_2 , taking $1/s' + 1/s = 1, 1/s = 1/r - \delta/n$, by using the Hölder's inequality and the boundedness of $g_{\mu,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$, we get

$$\begin{aligned}
V_2 &\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma| |g_{\mu,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)| dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{s'} dx \right)^{1/s'} \times \\
&\quad \times \left(\frac{1}{|Q|} \int_{R^n} |g_{\mu,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f \chi_Q)(x)|^s dx \right)^{1/s} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} |Q|^{-1/s} \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{s'} dx \right)^{1/s'} \times \\
&\quad \times \left(\int_{R^n} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c} f(x) \chi_Q(x)|^r dx \right)^{1/r} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} |Q|^{-1/s+1/r} \|\vec{b}_\sigma\|_{\text{Lip}_\beta} |Q|^{|\sigma|\beta/n} \|\vec{b}_{\sigma^c}\|_{\text{Lip}_\beta} |Q|^{|\sigma^c|\beta/n} \times \\
&\quad \times \left(\frac{1}{|Q|} \int_Q |f(x)|^r dx \right)^{1/r} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} \left(\frac{1}{|Q|^{1-(m\beta+\delta)r/n}} \int_Q |f(x)|^r dx \right)^{1/r} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} M_{m\beta+\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For V_3 , by the boundness of $g_{\mu,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$ and Hölder's inequality we get

$$\begin{aligned}
V_3 &\leq C \left(\frac{1}{|Q|} \int_{R^n} |g_{\mu,\delta}(\prod_{j=1}^m (b_j - (b_j)_Q) f \chi_Q)(x)|^s dx \right)^{1/s} \\
&\leq C |Q|^{-1/s} \left(\int_{R^n} |\prod_{j=1}^m (b_j(x) - (b_j)_Q)|^r |f(x) \chi_Q(x)|^r dx \right)^{1/r} \\
&\leq C |Q|^{-1/s} |Q|^{m\beta/n} |Q|^{1/r} \|\vec{b}\|_{Lip_\beta} \left(\frac{1}{|Q|} \int_Q |f(x)|^r dx \right)^{1/r} \\
&\leq C \|\vec{b}\|_{Lip_\beta} M_{m\beta+\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For V_4 , similar to the proof of IV in (a), we get

$$\begin{aligned}
&\int_{(Q)^c} \prod_{j=1}^m |b_j(z) - (b_j)_Q| |f(z)| |x - x_0|^{1/2} \times \\
&\quad \times \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{t^{-n} dy dt}{(t + |y - z|)^{2n+2-2\delta}} \right)^{1/2} dz \\
&\leq C \int_{(Q)^c} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} |f(z)| \prod_{j=1}^m |b_j(z) - (b_j)_Q| dz \\
&\leq C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^k Q|^{1-\delta/n}} \int_{2^k Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_Q) \right|^{r'} dy \right)^{1/r'} \times \\
&\quad \times \left(\frac{1}{|2^k Q|^{1-\delta/n}} \int_{2^k Q} |f(y)|^r dy \right)^{1/r} \\
&\leq C \left(\frac{1}{|2^k Q|^{1-(m\beta+\delta)r/n}} \int_{2^k Q} |f(y)|^r dy \right)^{1/r} \|\vec{b}\|_{Lip_\beta} \\
&\leq C \|\vec{b}\|_{Lip_\beta} M_{m\beta+\delta,r}(f)(\tilde{x}).
\end{aligned}$$

So we have

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q |g_{\mu,\delta}^{\vec{b}}(f)(x) - g_{\mu,\delta}(((b_1)_Q - b_1) \dots ((b_m)_Q - b_m) f_2)(x_0)| dx \\
&\leq C \|\vec{b}\|_{Lip_\beta} M_{m\beta+\delta,r}(f)(\tilde{x}).
\end{aligned}$$

Thus,

$$(g_{\mu,\delta}^{\vec{b}}(f))^{\#} \leq C \|\vec{b}\|_{Lip_\beta} M_{m\beta+\delta,r}(f).$$

By using Lemma 3 and the boundedness of $g_{\mu,\delta}$, we have

$$\begin{aligned} \|g_{\mu,\delta}^{\vec{b}}(f)\|_{L^q} &\leq C\|(g_{\mu,\delta}^{\vec{b}}(f))^{\#}\|_{L^q} \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta}\|M_{m,\beta+\delta,r}(f)\|_{L^q} \leq C\|\vec{b}\|_{\text{Lip}_\beta}\|f\|_{L^p}. \end{aligned}$$

This completes the proof of (b) and the theorem. $\square$

Theorem 2. *Let $0 < \beta \leq 1$, $\mu > 3 + 1/n - 2\delta/n$, $n/(n + m\beta) < p \leq 1$, $1/q = 1/p - (m\beta + \delta)/n$, $b_j \in \text{Lip}_\beta(R^n)$ for $1 \leq j \leq m$. Then $g_{\mu,\delta}^{\vec{b}}$ is bounded from $H^p(R^n)$ to $L^q(R^n)$.*

Proof. It suffices to show that there exists a constant $C > 0$ such that for every H^p -atom a ,

$$\|g_{\mu,\delta}^{\vec{b}}(a)\|_{L^q} \leq C.$$

Let a be a H^p -atom, that is that a supported on a cube $Q = Q(x_0, r)$, $\|a\|_{L^\infty} \leq |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \leq [n(1/p - 1)]$.

When $m = 1$ see [9]. Now, consider the case $m \geq 2$. Write

$$\begin{aligned} \|g_{\mu,\delta}^{\vec{b}}(a)(x)\|_{L^q} &\leq \left(\int_{|x-x_0| \leq 2r} |g_{\mu,\delta}^{\vec{b}}(a)(x)|^q dx \right)^{1/q} \\ &\quad + \left(\int_{|x-x_0| > 2r} |g_{\mu,\delta}^{\vec{b}}(a)(x)|^q dx \right)^{1/q} = I + II. \end{aligned}$$

For I , choose $1 < p_1 < n/(m\beta + \delta)$ and q_1 such that $1/q_1 = 1/p_1 - (m\beta + \delta)/n$. By the boundedness of $g_{\mu,\delta}^{\vec{b}}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$ (see Theorem 1), the size condition of a and Hölder's inequality, we get

$$\begin{aligned} I &\leq C\|g_{\mu,\delta}^{\vec{b}}(a)\|_{L^{q_1}} r^{n(1/q-1/q_1)} \leq C\|\vec{b}\|_{\text{Lip}_\beta}\|a\|_{L^{p_1}} r^{n(1/q-1/q_1)} \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta} r^{n(-1/p+1/p_1)} r^{n(1/q-1/q_1)} \leq C\|\vec{b}\|_{\text{Lip}_\beta}. \end{aligned}$$

For II , let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \neq 0$. We get

$$\begin{aligned} |F_t^{\vec{b}}(a)(x, y)| &\leq \\ &\leq |(b_1(x) - b_1(x_0)) \dots (b_m(x) - b_m(x_0)) \int_B (\psi_t(y - z) - \psi_t(y - x_0)) a(z) dz| \\ &\quad + \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c} \int_B (\vec{b}(z) - \vec{b}(x_0))_{\sigma} \psi_t(y - z) a(z) dz| \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta} |x - x_0|^{m\beta} \cdot \int_B |\psi_t(y - z) - \psi_t(y - x_0)| |a(z)| dz \\ &\quad + C\|\vec{b}\|_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} |x - x_0|^{\tau\beta} \int_B |z - x_0|^{\tau'\beta} |\psi_t(y - z)| |a(z)| dz \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta} \frac{|x - x_0|^{m\beta} t}{(t + |y - x_0|)^{n+2-\delta}} \int_B |x_0 - z| |a(z)| dz \end{aligned}$$

$$\begin{aligned}
& +C||\vec{b}||_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \frac{t}{(t+|y-x_0|)^{n+1-\delta}} \int_B |z-x_0|^{\tau'\beta} |a(z)| dz \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} \frac{t}{(t+|y-x_0|)^{n+2-\delta}} \cdot r^{1+n(1-1/p)} \cdot |x-x_0|^{m\beta} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} \frac{t}{(t+|y-x_0|)^{n+1-\delta}} \cdot \sum_{\tau+\tau'=m} r^{\tau'\beta+n(1-1/p)} \cdot |x-x_0|^{\tau\beta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& |g_{\mu,\delta}^{\vec{b}}(a)(x)| \leq \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} \left(\int_0^\infty \left(\frac{t}{(t+|x-x_0|)^{n+2-\delta}} \right)^2 \frac{dt}{t} \right)^{1/2} \cdot r^{1+n(1-1/p)} \cdot |x-x_0|^{m\beta} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} \left(\int_0^\infty \left(\frac{t}{(t+|x-x_0|)^{n+1-\delta}} \right)^2 \frac{dt}{t} \right)^{1/2} \times \\
& \quad \times \sum_{\tau+\tau'=m} r^{\tau'\beta+n(1-1/p)} \cdot |x-x_0|^{\tau\beta} \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} |x-x_0|^{-(n+1-\delta)} \cdot r^{1+n(1-1/p)} \cdot |x-x_0|^{m\beta} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} |x-x_0|^{-(n-\delta)} \cdot \sum_{\tau+\tau'=m} r^{\tau'\beta+n(1-1/p)} \cdot |x-x_0|^{\tau\beta},
\end{aligned}$$

so

$$\begin{aligned}
II & \leq C||\vec{b}||_{\text{Lip}_\beta} \cdot r^{1+n(1-1/p)} \left(\int_{|x-x_0|>2r} |x-x_0|^{-(n+1-\delta-m\beta)q} dx \right)^{1/q} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} \cdot \sum_{\tau+\tau'=m} r^{\tau'\beta+n(1-1/p)} \left(\int_{|x-x_0|>2r} |x-x_0|^{-(n-\delta-\tau\beta)q} dx \right)^{1/q} \\
& = J_1 + J_2,
\end{aligned}$$

we get

$$\begin{aligned}
J_1 & \leq C||\vec{b}||_{\text{Lip}_\beta} \cdot r^{1+n(1-1/p)} \left(\sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^kQ} (2^k r)^{-(n+1-\delta-m\beta)q} dx \right)^{1/q} \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} \cdot r^{1+n(1-1/p)} \left(\sum_{k=1}^\infty 2^{-kq(n+1-\delta-m\beta)} r^{-q(n+1-\delta-m\beta)} 2^{(k+1)n} r^n dx \right)^{1/q} \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} \sum_{k=1}^\infty 2^{-k(n+1-\delta-m\beta-n/q)} r^{1+n(1-1/p)-(n+1-\delta-m\beta)+n/q} \\
& \leq C||\vec{b}||_{\text{Lip}_\beta}.
\end{aligned}$$

For J_2 , similar to J_1 , we have $J_2 \leq C\|\vec{b}\|_{\text{Lip}_\beta}$.

Combining the estimates for I and II , then leads to the desired result. $\square$

Theorem 3. *Let $0 < \beta \leq 1$, $\mu > 3 + 1/n - 2\delta/n$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = (m\beta + \delta)/n$, $n(1 - 1/q_1) \leq \alpha < n(1 - 1/q_1) + \varepsilon$, $\varepsilon < \min(1, m\beta)$, $b_j \in \text{Lip}_\beta(R^n)$ for $1 \leq j \leq m$. Then $g_{\mu,\delta}^{\vec{b}}$ is bounded from $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ to $\dot{K}_{q_2}^{\alpha,p}(R^n)$.*

Proof. By Lemma 7, let $f \in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, $\text{supp } a_j \subset B_j = B(0, 2^j)$, a_j be a central (α, q_1) -atom, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$.

$$\begin{aligned} \|g_{\mu,\delta}^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha,p}}^p &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|g_{\mu,\delta}^{\vec{b}}(a_j) \chi_k\|_{L^{q_2}} \right)^p \\ &\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|g_{\mu,\delta}^{\vec{b}}(a_j) \chi_k\|_{L^{q_2}} \right)^p = I + II. \end{aligned}$$

For II , by the boundedness of $g_{\mu,\delta}^{\vec{b}}$ on (L^{q_1}, L^{q_2}) (see Theorem 1), we have

$$\begin{aligned} II &\leq C\|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1}} \right)^p \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \cdot 2^{(k-j)\alpha} \right)^p \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta}^p \begin{cases} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p}, & 0 < p \leq 1 \\ \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j+1} |\lambda_j|^p \cdot 2^{(k-j)\alpha p/2} \right) \left(\sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p'/2} \right)^{p/p'}, & 1 < p < \infty \end{cases} \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

For I , we have

$$\begin{aligned} &|F_t^{\vec{b}}(a_j)(x, y)| \leq \\ &\leq |(b_1(x) - b_1(0)) \cdots (b_m(x) - b_m(0)) \int_{B_j} (\psi_t(y - z) - \psi_t(y)) a_j(z) dz| \\ &\quad + \sum_{j=1}^{\infty} \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(0))_{\sigma^c} \int_{B_j} (\vec{b}(z) - \vec{b}(0))_{\sigma} \psi_t(y - z) a_j(z) dz| \\ &\leq C\|\vec{b}\|_{\text{Lip}_\beta} |x|^{m\beta} \int_{B_j} |\psi_t(y - z) - \psi_t(y)| |a_j(z)| dz \end{aligned}$$

$$\begin{aligned}
& +C||\vec{b}||_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \int_{B_j} |z|^{\tau'\beta} |\psi_t(y-z)| |a_j(z)| dz \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} \frac{|x|^{m\beta} t}{(t+|y|)^{n+2-\delta}} \cdot 2^{j(1+n(1-1/q_1)-\alpha)} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} \frac{|x|^{\tau\beta} t}{(t+|y|)^{n+1-\delta}} \cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& g_{\mu,\delta}^{\vec{b}}(a_j)(x) \leq \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} |x|^{m\beta} \cdot 2^{j(1+n(1-1/q_1)-\alpha)} \cdot \left(\int_0^\infty \left(\frac{t}{(t+|x|)^{n+2-\delta}} \right)^2 \frac{dt}{t} \right)^{1/2} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)} \times \\
& \quad \times \left(\int_0^\infty \left(\frac{t}{(t+|x|)^{n+1-\delta}} \right)^2 \frac{dt}{t} \right)^{1/2} \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} |x|^{m\beta} |x|^{-(n+1-\delta)} \cdot 2^{j(1+n(1-1/q_1)-\alpha)} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} |x|^{-(n-\delta)} \cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)},
\end{aligned}$$

and

$$\begin{aligned}
& ||g_{\mu,\delta}^{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \leq \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} \cdot 2^{j(1+n(1-1/q_1)-\alpha)} \cdot \left(\int_{B_k \setminus B_{k-1}} |x|^{-(n-(m\beta+\delta)+1)q_2} dx \right)^{1/q_2} \\
& \quad +C||\vec{b}||_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)} \cdot \left(\int_{B_k \setminus B_{k-1}} |x|^{-(n-\tau\beta-\delta)q_2} dx \right)^{1/q_2} \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} (2^{j(1+n(1-1/q_1)-\alpha)-k(1+n(1-1/q_1))} \\
& \quad + \sum_{\tau+\tau'=m} 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)-k(\tau'\beta+n(1-1/q_1))}) \\
& \leq C||\vec{b}||_{\text{Lip}_\beta} 2^{-k\alpha} (2^{(j-k)(1+n(1-1/q_1)-\alpha)} + 2^{(j-k)(m\beta+n(1-1/q_1)-\alpha)}),
\end{aligned}$$

so

$$\begin{aligned}
I &\leq C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \cdot 2^{(j-k)(1+n(1-1/q_1)-\alpha)} \right)^p \\
&\quad + C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \cdot 2^{(j-k)(m\beta+n(1-1/q_1)-\alpha)} \right)^p.
\end{aligned}$$

When $0 < p \leq 1$,

$$\begin{aligned}
I &\leq C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{p(j-k)(1+n(1-1/q_1)-\alpha)} \right) \\
&\quad + C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{p(j-k)(m\beta+n(1-1/q_1)-\alpha)} \right) \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta}^p \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{p(j-k)(1+n(1-1/q_1)-\alpha)} \right. \\
&\quad \left. + \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{p(j-k)(m\beta+n(1-1/q_1)-\alpha)} \right) \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned}$$

When $p > 1$,

$$\begin{aligned}
I &\leq C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{p(j-k)(1+n(1-1/q_1)-\alpha)/2} \right) \times \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} 2^{p'(j-k)(1+n(1-1/q_1)-\alpha)/2} \right)^{p/p'} \\
&\quad + C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{p(j-k)(m\beta+n(1-1/q_1)-\alpha)/2} \right) \times \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} 2^{p'(j-k)(m\beta+n(1-1/q_1)-\alpha)/2} \right)^{p/p'} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned}$$

From I and II , we have

$$\|g_{\mu,\delta}^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha,p}} \leq C \|\vec{b}\|_{\text{Lip}_\beta} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{H\dot{K}_{q_1}^{\alpha,p}}$$

This completes the proof of Theorem 3. $\square$

When $\alpha = n(1 - 1/q_1) + \varepsilon$, $\varepsilon < \min(1, m\beta)$, this kind of boundedness fails. Now, we give an estimate of weak type.

Theorem 4. *Let $0 < \beta \leq 1$, $0 < p \leq 1$, $1 < q_1, q_2 < \infty$, $1/q_2 = 1/q_1 - (m\beta + \delta)/n$, $b_j \in \text{Lip}_\beta(R^n)$ for $1 \leq j \leq m$. Then $g_{\mu, \delta}^{\vec{b}}$ maps $H\dot{K}_{q_1}^{n(1-1/q_1)+\varepsilon, p}(R^n)$ continuously into $W\dot{K}_{q_2}^{n(1-1/q_1)+\varepsilon, p}(R^n)$, where $0 < \varepsilon < \min(1, m\beta)$.*

Proof. We write $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$, where each a_k is a central $(n(1-1/q_1)+\varepsilon, q_1)$ atom supported on B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Write

$$\begin{aligned} & \|g_{\mu, \delta}^{\vec{b}}\|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\varepsilon, p}} \leq \\ & \leq \sup_{\lambda > 0} \lambda \left\{ \sum_{l=-\infty}^{\infty} 2^{l(n(1-1/q_1)+\varepsilon)p} |\{x \in E_l : |g_{\mu, \delta}^{\vec{b}}(\sum_{k=l-3}^{\infty} \lambda_k a_k)(x)| > \lambda/2\}|^{p/q_2} \right\}^{1/p} \\ & + \sup_{\lambda > 0} \lambda \left\{ \sum_{l=-\infty}^{\infty} 2^{l(n(1-1/q_1)+\varepsilon)p} |\{x \in E_l : |g_{\mu, \delta}^{\vec{b}}(\sum_{k=-\infty}^{l-4} \lambda_k a_k)(x)| > \lambda/2\}|^{p/q_2} \right\}^{1/p} \\ & = G_1 + G_2. \end{aligned}$$

By the (L^{q_1}, L^{q_2}) boundedness of $g_{\mu, \delta}^{\vec{b}}$ and an estimate similar to that for I_1 in Theorem 3, we get

$$G_1^p \leq C \sum_{l=-\infty}^{\infty} 2^{lp(n(1-1/q_1)+\varepsilon)} \|g_{\mu, \delta}^{\vec{b}}(\sum_{k=l-3}^{\infty} \lambda_k a_k)(x) \chi_l\|_{q_2}^p \leq C \|\vec{b}\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} |\lambda_k|^p.$$

To estimate G_2 , let us now use the estimate

$$\begin{aligned} g_{\mu, \delta}^{\vec{b}}(a_k)(x) & \leq C \|\vec{b}\|_{\text{Lip}_\beta} |x|^{m\beta} |x|^{-(n+1-\delta)} \cdot 2^{k(1+n(1-1/q_1)-\alpha)} \\ & + C \|\vec{b}\|_{\text{Lip}_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} |x|^{-(n-\delta)} \cdot 2^{k(\tau'\beta+n(1-1/q_1)-\alpha)}, \end{aligned}$$

which we get in the proof of Theorem 3. Note that when $x \in E_l$,

$$\alpha = n(1 - 1/q_1) + \varepsilon,$$

$$\begin{aligned} \lambda & < 2 \sum_{k=-\infty}^{l-4} |\lambda_k| |g_{\mu, \delta}^{\vec{b}}(a_k)(x)| \leq \\ & \leq C \|\vec{b}\|_{\text{Lip}_\beta} \sum_{k=-\infty}^{l-4} |\lambda_k| (2^l)^{m\beta+\delta-n-1} \sum_{k=-\infty}^{l-4} (2^k)^{1+n(1-1/q_1)-\alpha} \\ & + C \|\vec{b}\|_{\text{Lip}_\beta} \sum_{k=-\infty}^{l-4} |\lambda_k| \sum_{\tau+\tau'=m} (2^l)^{\tau\beta+\delta-n} \sum_{k=-\infty}^{l-4} (2^k)^{\tau'\beta+n(1-1/q_1)-\alpha} \end{aligned}$$

$$\begin{aligned}
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} \sum_{k=-\infty}^{l-4} |\lambda_k| (2^l)^{m\beta+\delta-n-\varepsilon} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} 2^{l(m\beta+\delta-n-\varepsilon)} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p},
\end{aligned}$$

for $\lambda > 0$, let l_λ be the maximal positive integer satisfying

$$2^{l_\lambda(n+\varepsilon-m\beta-\delta)} \leq C \|\vec{b}\|_{\text{Lip}_\beta} \lambda^{-1} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p},$$

then if $l > l_\lambda$, we have

$$|\{x \in E_l : |g_{\mu,\delta}^{\vec{b}}(\sum_{k=-\infty}^{l-4} \lambda_k a_k)| > \lambda/2\}| = 0,$$

so we obtain

$$\begin{aligned}
G_2 &\leq \sup_{\lambda>0} \lambda \left\{ \sum_{l=-\infty}^{l_\lambda} 2^{l(n(1-1/q_1)+\varepsilon)p} (2^l)^{np/q_2} \right\}^{1/p} \\
&\leq \sup_{\lambda>0} \lambda \left\{ \sum_{l=-\infty}^{l_\lambda} (2^l)^{(n+\varepsilon-m\beta-\delta)} \right\} \\
&\leq \sup_{\lambda>0} \lambda 2^{l_\lambda(n+\varepsilon-m\beta-\delta)} \\
&\leq C \|\vec{b}\|_{\text{Lip}_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}.
\end{aligned}$$

Now, combining the above estimates for G_1 and G_2 , we obtain

$$\|g_{\mu,\delta}^{\vec{b}}(f)\|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\varepsilon,p}} \leq C \|\vec{b}\|_{\text{Lip}_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}.$$

Theorem 4 follows by taking the infimum over all central atomic decompositions. $\square$

ACKNOWLEDGEMENT

The authors would like to express their gratitude to the referee for his comments.

REFERENCES

- [1] J. Álvarez, R. J. Bagby, D. S. Kurtz, and C. Pérez. Weighted estimates for commutators of linear operators. *Studia Math.*, 104(2):195–209, 1993.
- [2] S. Chanillo. A note on commutators. *Indiana Univ. Math. J.*, 31(1):7–16, 1982.
- [3] W. Chen. A Besov estimate for multilinear singular integrals. *Acta Math. Sin. (Engl. Ser.)*, 16(4):613–626, 2000.

- [4] R. R. Coifman, R. Rochberg, and G. Weiss. Factorization theorems for Hardy spaces in several variables. *Ann. of Math. (2)*, 103(3):611–635, 1976.
- [5] R. A. DeVore and R. C. Sharpley. Maximal functions measuring smoothness. *Mem. Amer. Math. Soc.*, 47(293):viii+115, 1984.
- [6] J. García-Cuerva and M.-J. L. Herrero. A theory of Hardy spaces associated to the Herz spaces. *Proc. London Math. Soc. (3)*, 69(3):605–628, 1994.
- [7] G. Hu, S. Lu, and D. Yang. The weak Herz spaces. *Beijing Shifan Daxue Xuebao*, 33(1):27–34, 1997.
- [8] S. Janson. Mean oscillation and commutators of singular integral operators. *Ark. Mat.*, 16(2):263–270, 1978.
- [9] L. Liu. Boundedness for multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces. *Extracta Math.*, 19(2):243–255, 2004.
- [10] L. Liu. Boundedness of multilinear operators on Triebel-Lizorkin spaces. *Int. J. Math. Math. Sci.*, (5-8):259–271, 2004.
- [11] L. Liu. The continuity of commutators on Triebel-Lizorkin spaces. *Integral Equations Operator Theory*, 49(1):65–75, 2004.
- [12] S. Lu, Q. Wu, and D. Yang. Boundedness of commutators on Hardy type spaces. *Sci. China Ser. A*, 45(8):984–997, 2002.
- [13] S. Z. Lu. *Four lectures on real H^p spaces*. World Scientific Publishing Co. Inc., River Edge, NJ, 1995.
- [14] S. Z. Lu and D. C. Yang. The decomposition of weighted Herz space on $\mathbf{R}^n$ and its applications. *Sci. China Ser. A*, 38(2):147–158, 1995.
- [15] S. Z. Lu and D. C. Yang. The weighted Herz-type Hardy space and its applications. *Sci. China Ser. A*, 38(6):662–673, 1995.
- [16] M. Paluszyński. Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. *Indiana Univ. Math. J.*, 44(1):1–17, 1995.
- [17] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
- [18] A. Torchinsky. *Real-variable methods in harmonic analysis*, volume 123 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.

Received June 3, 2008.

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