

## NOTE ON AN INEQUALITY OF F. QI AND L. DEBNATH

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ABSTRACT. In this paper, a similar result of F. Qi and L. Debnath's inequality is given, and a generalization of Alzer's inequality is established.

### 1. INTRODUCTION

It is well-known that the following inequality

$$(1.1) \quad \frac{n}{n+1} < \left( \frac{(1/n) \sum_{i=1}^n i^r}{(1/(n+1)) \sum_{i=1}^{n+1} i^r} \right)^{1/r} \leq \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}$$

holds for  $r > 0$  and  $n \in \mathbb{N}$ . We call the left-hand side of this inequality Alzer's inequality [2], and the right-hand side Martins's inequality [4].

Alzer's inequality has invoked the interest of several mathematicians, we refer the reader to [3, 4, 5, 6, 7] and the references therein.

In [6] F. Qi and L. Debnath gave a further generalization of (1.1), they proved the following result:

**Theorem 1.1.** *Let  $n$  and  $m$  be natural numbers. Suppose  $\{a_1, a_2, \dots\}$  is a positive and increasing sequence satisfying*

$$(1.2) \quad \frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - ka_k^r} \geq \left( \frac{a_{k+2}}{a_{k+1}} \right)^r$$

for any given positive real number  $r$  and  $k \in \mathbb{N}$ . Then we have the inequality

$$(1.3) \quad \frac{a_n}{a_{n+m}} \leq \left( \frac{(1/n) \sum_{i=1}^n a_i^r}{(1/(n+m)) \sum_{i=1}^{n+m} a_i^r} \right)^{1/r}.$$

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Chen and F. Qi in [3] show that Alzer's inequality (1.1) is valid for  $r < 0$ .

Motivated by approach of [3], a natural question is does (1.3) still hold for  $r < 0$ . In this paper, we show that (1.3) is no longer valid for  $r < 0$ . But, we found another similar result.

## 2. MAIN RESULTS

**Theorem 2.1.** *Let  $n$  and  $m$  be natural numbers. Suppose  $\{a_1, a_2, \dots\}$  is a positive and decreasing sequence satisfying*

$$(2.1) \quad \frac{(k+2)a_{k+1}^s - (k+1)a_{k+2}^s}{(k+1)a_k^s - ka_{k+1}^s} \geq \left(\frac{a_{k+1}}{a_k}\right)^s,$$

for any given positive real number  $s$  and  $k \in \mathbb{N}$ . Then we have the inequality

$$(2.2) \quad \frac{a_{m+n}}{a_n} \leq \left( \frac{(1/n) \sum_{i=1}^n \frac{1}{a_i^s}}{(1/(n+m)) \sum_{i=1}^{n+m} \frac{1}{a_i^s}} \right)^{1/s}.$$

The lower bound of (2.2) is best possible.

*Proof.* The inequality (2.2) is equivalent to

$$(2.3) \quad \frac{a_{n+m}^s}{a_n^s} \leq \frac{\frac{1}{n} \sum_{i=1}^n \frac{1}{a_i^s}}{\frac{1}{n+m} \sum_{i=1}^{n+m} \frac{1}{a_i^s}}.$$

This is also equivalent to

$$(2.4) \quad \frac{a_{n+1}^s}{a_n^s} \leq \frac{\frac{1}{n} \sum_{i=1}^n \frac{1}{a_i^s}}{\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{a_i^s}}.$$

That is,

$$(2.5) \quad \frac{a_{n+1}^s}{(n+1)} \sum_{i=1}^{n+1} \frac{1}{a_i^s} \leq \frac{a_n^s}{n} \sum_{i=1}^n \frac{1}{a_i^s}.$$

Since,

$$(2.6) \quad \sum_{i=1}^{n+1} \frac{1}{a_i^s} = \sum_{i=1}^n \frac{1}{a_i^s} + \frac{1}{a_{n+1}^s}.$$

Inequality (2.5) reduces to

$$(2.7) \quad \sum_{i=1}^n \frac{1}{a_i^s} \geq \frac{n}{(n+1)a_n^s - na_{n+1}^s}.$$

Since,  $\{a_1, a_2, \dots\}$  is a positive and decreasing sequence, it is easy to see that inequality (2.7) holds for  $n = 1$ .

Assume that (2.7) holds for  $n > 1$ . Using the principle of induction, it is easy to show that (2.7) holds for  $n+1$ . Using equality (2.6), the induction can be written as (2.1) for  $k = n$ . Thus, inequality (2.7) holds.

It can easily be shown that

$$(2.8) \quad \lim_{s \rightarrow +\infty} \left( \frac{(1/n) \sum_{i=1}^n \frac{1}{a_i^s}}{(1/(n+m)) \sum_{i=1}^{n+m} \frac{1}{a_i^s}} \right)^{1/s} = \frac{a_{m+n}}{a_n}$$

Hence, the lower bound of (2.2) is best possible. The proof is complete.  $\square$

The following example shows that the sequence satisfying (2.1) exists.

*Example 2.2.* Let  $a_k = \frac{1}{k}$ , ( $k = 1, 2, \dots$ ), then

$$\frac{(k+2)a_{k+1}^s - (k+1)a_{k+2}^s}{(k+1)a_k^s - ka_{k+1}^s} = \frac{k^s}{(k+2)^s} \cdot \frac{(k+2)^{s+1} - (k+1)^{s+1}}{(k+1)^{s+1} - k^{s+1}}$$

Define function  $f, g: [k, k+1] \rightarrow R$ , where  $f(x) = (x+1)^{s+1}$ ,  $g(x) = x^{s+1}$ ,  $s > 0$ . Applying the Cauchy's mean-value theorem, it turns out that there exists one point  $\xi \in (k, k+1)$  such that

$$\begin{aligned} \frac{k^s}{(k+2)^s} \cdot \frac{(k+2)^{s+1} - (k+1)^{s+1}}{(k+1)^{s+1} - k^{s+1}} &= \frac{k^s}{(k+2)^s} \cdot \frac{f(k+1) - f(k)}{g(k+1) - g(k)} \\ &= \frac{k^s}{(k+2)^s} \cdot \frac{f'(\xi)}{g'(\xi)} \\ &= \frac{k^s}{(k+2)^s} \cdot \left( \frac{1+\xi}{\xi} \right)^s \\ &\geq \frac{k^s}{(k+2)^s} \cdot \left( 1 + \frac{1}{k+1} \right)^s \\ &= \left( \frac{k}{k+1} \right)^s = \left( \frac{a_{k+1}}{a_k} \right)^s \end{aligned}$$

Hence,

$$\frac{(k+2)a_{k+1}^s - (k+1)a_{k+2}^s}{(k+1)a_k^s - ka_{k+1}^s} \geq \left( \frac{a_{k+1}}{a_k} \right)^s$$

Let  $a_k = \frac{1}{k}$  in Theorem 2.1, then we have

**Corollary 2.1.**

$$(2.9) \quad \frac{n}{n+m} < \left( \frac{(1/n) \sum_{i=1}^n i^s}{(1/(n+m)) \sum_{i=1}^{n+m} i^r} \right)^{1/s},$$

where  $s > 0$  and  $m, n \in \mathbb{N}$ .

Let  $m = 1$  in (2.9), then we get (1.1).

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