

ON DIFFERENTIAL SANDWICH THEOREMS FOR SOME SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS

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ABSTRACT. In this present investigation we study certain application of differential subordination and superordination for the class of multivalent functions to be subordinated and superordinated by convex functions.

1. INTRODUCTION

Let $\mathcal{H}$ be the class of analytic functions in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be a subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$(1.1) \quad f(z) := z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \Delta),$$

and let $\mathcal{A} := \mathcal{A}_1$. Komatu [4] introduced the family of integral operators, defined by

$$(1.2) \quad I_a^\sigma f(z) := \frac{(1+a)^\sigma}{z^a \Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} t^{a-1} f(t) dt,$$

where $a > -1, \sigma > 0$ and $f \in \mathcal{A}$. It can be easily observed that

$$(1.3) \quad I_a^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right) a_n z^n.$$

From (1.2) and (1.3) it can be seen that

$$z(I_a^{\sigma+1} f(z))' = (1+a)I_a^\sigma f(z) - aI_a^{\sigma+1} f(z).$$

Let $p, h \in \mathcal{H}$ and let

$$\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C}.$$

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If $p(z)$ and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if $p(z)$ satisfy the second order superordination

$$(1.4) \quad h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z),$$

then p is the solution of the differential superordination (1.4). (If f is subordinate to F , then we say F is superordinate to f). An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfy $q \prec \tilde{q}$ for all subordinants q of (1.4) is said to be best subordinant. Recently Miller and Mocanu [6] obtained conditions on h , q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [6], Bulboacă [3] have considered certain classes of first order differential subordinations as well as superordination preserving integral operators [2].

Over many years, several authors have studied the application of differential subordination and superordination for functionals like $\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, \frac{f(z)}{zf'(z)}$. Recently Obradović and Owa [7] obtained some subordination result in terms of $\left(\frac{f(z)}{z}\right)^\mu$.

In this present investigation we give some applications of first order differential subordination and superordination to obtain sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu \prec q_2(z)$$

where q_1 and q_2 are univalent in Δ . Interestingly various well known results are special cases of our results.

2. PRELIMINARIES

For the present investigation we need the following definition and results.

Definition 2.1. [6, Definition 2, p. 817] Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\Delta - E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z \rightarrow \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem 2.1 ([5, Theorem 3.4h, p. 132]). *Let q be univalent in the disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$.*

Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that

- (1) Q is starlike univalent in Δ and
- (2) $\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in \Delta$.

If ξ is analytic in Δ with $\xi(\Delta) \subseteq D$, and

$$(2.1) \quad \theta(\xi(z)) + z\xi'(z)\phi(\xi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $\xi \prec q$ and q is the best dominant.

Theorem 2.2 ([3]). Let q be univalent in the unit disk Δ and ϑ and ϕ be analytic in a domain D containing $q(\Delta)$, suppose that

- (1) $\Re \frac{\vartheta'q(z)}{\psi(q(z))} > 0$ for all $z \in \Delta$ and
- (2) $\xi(z) = zq'(z)\psi(q(z))$ is starlike univalent function in Δ .

If $\xi \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, with $\xi(\Delta) \subset D$, and $\vartheta(\xi(z)) + z\xi'(z)\psi(\xi(z))$ is univalent in Δ , and

$$(2.2) \quad \vartheta(q(z)) + zq'(z)\psi(q(z)) \prec \vartheta(\xi(z)) + z\xi'(z)\psi(\xi(z)),$$

then $q \prec \xi$ and q is the best subdominant.

Theorem 2.3 ([5, Lemma 1, p. 71]). Let h be convex univalent in Δ with $h(0) = a$ and $0 \neq \gamma \in \mathbb{C}$ and $\Re \gamma > 0$. If $p \in \mathcal{H}[a, n]$ and

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and is the best dominant.

3. APPLICATION OF DIFFERENTIAL SUBORDINATION

Theorem 3.1. Let α, β and γ be complex numbers with $\gamma \neq 0$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfy

$$(3.1) \quad \Re \left\{ \frac{\alpha + 2\beta q(z)}{\gamma} \right\} > 0.$$

Let $f \in \mathcal{A}_p$ and

$$(3.2) \quad \psi(z) := \frac{\alpha}{p} \left(\frac{f(z)}{z^p} \right)^\mu + \frac{\beta}{p^2} \left(\frac{f(z)}{z^p} \right)^{2\mu} + \gamma \mu \left(\frac{f(z)}{z^p} \right)^\mu \left[\frac{zf'(z)}{pf(z)} - 1 \right].$$

If

$$\psi(z) \prec \alpha q(z) + \beta q^2(z) + \gamma zq'(z)$$

then

$$\frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu \prec q(z)$$

and q is the best dominant.

Proof. Define the function $\xi(z)$ by

$$(3.3) \quad \xi(z) := \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu.$$

A computation using (3.3) shows that

$$\frac{z\xi'(z)}{\xi(z)} = \frac{z\mu f'(z)}{f(z)} - \mu p.$$

Also we find that

$$\begin{aligned} \psi(z) &:= \alpha\xi(z) + \beta\xi^2(z) + \gamma z\xi'(z) \\ &= \frac{\alpha}{p} \left(\frac{f(z)}{z^p} \right)^\mu + \frac{\beta}{p^2} \left(\frac{f(z)}{z^p} \right)^{2\mu} + \gamma\mu \left(\frac{f(z)}{z^p} \right)^\mu \left[\frac{zf'(z)}{pf(z)} - 1 \right]. \end{aligned}$$

Since $\psi(z) \prec \alpha q(z) + \beta q^2(z) + \gamma zq'(z)$, this can be written as (2.1), when $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \gamma$. Note that $\phi(w) \neq 0$ and $\theta(w), \phi(w)$ are analytic in $\mathbb{C}$. Set

$$\begin{aligned} Q(z) &:= \gamma zq'(z) \\ h(z) &:= \theta(q(z)) + Q(z) \\ &= \alpha q(z) + \beta q^2(z) + \gamma zq'(z). \end{aligned}$$

In light of the hypothesis of Theorem 2.1, we see that Q is starlike and

$$\Re \frac{zh'(z)}{Q(z)} = \Re \left\{ \frac{\alpha + 2\beta q(z)}{\gamma} + \left(1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0.$$

Hence the result follows as an application of Theorem 2.1. $\square$

Theorem 3.2.

(1) Let $0 \neq \delta \in \mathbb{C}$ and q be convex univalent in Δ with $q(0) = 1$ and satisfy

$$\Re \left\{ \frac{\mu}{\delta} \right\} > 0.$$

If $f \in \mathcal{A}$ satisfy

$$(1 - \delta) \left(\frac{f(z)}{z} \right)^\mu + \delta \left(f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) \prec q(z) + \frac{\delta}{\mu} zq'(z)$$

then

$$\left(\frac{f(z)}{z} \right)^\mu \prec q(z).$$

(2) If $f \in \mathcal{A}$ satisfy

$$f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} - \left(\frac{f(z)}{z} \right)^\mu \prec \frac{1}{\mu} zq'(z)$$

then

$$\left(\frac{f(z)}{z} \right)^\mu \prec q(z)$$

and q is the best dominant.

Proof. Proof of the first part follows from Theorem 3.1, by taking $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{\delta}{\mu}$.

The proof of the second part follows from Theorem 3.1, by taking $\alpha = \beta = 0, p = 1$ and $\gamma = \frac{1}{\mu}$. $\square$

By taking $\delta = \mu = n$ and $q(z) = \beta + (1 - \beta) \left[-1 - \frac{2}{z} \log(1 - z) \right]$ in first part of Theorem 3.2, we get the following result of Ponnusamy [8].

Corollary 3.3. *Let $f \in \mathcal{A}$. Then for a positive integer n , we have*

$$\Re \left\{ (1 - n) \left(\frac{f(z)}{z} \right)^n + n f'(z) \left(\frac{f(z)}{z} \right)^{n-1} \right\} > \beta$$

implies

$$\left(\frac{f(z)}{z} \right)^n \prec \beta + (1 - \beta) \left(-1 - \frac{2}{z} \log(1 - z) \right)$$

and $\beta + (1 - \beta) \left(-1 - \frac{2}{z} \log(1 - z) \right)$ is the best dominant.

By taking $\mu = 1$ and $q(z) = 1 + \left(\frac{A}{1 + \delta} \right) z$ in Theorem 3.2 and $\mu = \delta = 1$ and $q(z) = \frac{A}{B} + \left(1 - \frac{A}{B} \right) \frac{\log(1 + Bz)}{Bz}$ where δ, A and B are non zero complex numbers and $\Re \delta > 0$ and $-1 \leq B < A \leq 1$ in Theorem 3.2 we get the following result of Ponnusamy and Juneja [9].

Corollary 3.4. *Let $f \in \mathcal{A}$. Let δ be a complex number with $\Re \delta \geq 0$ and $-1 \leq B < A \leq 1$. Then*

$$(1 - \delta) \frac{f(z)}{z} + \delta f'(z) \prec 1 + Az$$

implies

$$\frac{f(z)}{z} \prec 1 + \left(\frac{A}{1 + \delta} \right) z$$

and the function $1 + \left(\frac{A}{1 + \delta} \right) z$ is the best dominant. Also

$$f'(z) \prec \frac{1 + Az}{1 + Bz}$$

implies

$$\frac{f(z)}{z} \prec \frac{A}{B} + \left(1 - \frac{A}{B} \right) \frac{\log(1 + Bz)}{Bz}$$

and the function $\frac{A}{B} + \left(1 - \frac{A}{B} \right) \frac{\log(1 + Bz)}{Bz}$ is the best dominant.

By taking $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{1}{\mu}$ in Theorem 3.1, we have the following result:

Corollary 3.5. *If $f \in \mathcal{A}$ satisfy*

$$f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \prec q(z) + \frac{zq'(z)}{\mu}$$

implies

$$\left(\frac{f(z)}{z} \right)^{\mu} \prec q(z)$$

and q is the best dominant.

Theorem 3.6. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \neq 0$. Let q be convex univalent in Δ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ . Further assume that*

$$(3.4) \quad \Re \left\{ \frac{\beta q(z)}{\gamma} - \frac{zq'(z)}{q(z)} \right\} > 0.$$

Let $f \in \mathcal{A}_p$ and if

$$\alpha + \frac{\beta}{p} \left(\frac{f(z)}{z^p} \right)^{\mu} + \gamma \mu \left[\frac{zf'(z)}{f(z)} - p \right] \prec \alpha + \beta q(z) + \frac{\gamma zq'(z)}{q(z)}$$

then

$$\frac{1}{p} \left(\frac{f(z)}{z^p} \right)^{\mu} \prec q(z)$$

where q is the best dominant.

Proof. Let $\theta(w) := \alpha + \beta w$ and $\phi(w) := \frac{\gamma}{w}$. Note that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Hence the result follows as an application of Theorem 2.1 for $\xi(z) := \frac{1}{p} \left(\frac{f(z)}{z} \right)^{\mu}$. $\square$

By taking $\alpha = p = 1, \beta = 0, \gamma = \frac{1}{\mu}$ and $q(z) = e^{\lambda Az}$, in Theorem 3.6 we get the following result obtained by Obradović and Owa [7].

Corollary 3.7. *Let $f \in \mathcal{A}$. If*

$$\frac{zf'(z)}{f(z)} \prec 1 + Az$$

then

$$\left(\frac{f(z)}{z} \right)^{\mu} \prec e^{\lambda Az},$$

where $e^{\lambda Az}$ is the best dominant.

We remark here that $q(z) = e^{\lambda Az}$ is univalent if and only if $|\lambda A| < \pi$.

For a special case when $q(z) = \frac{1}{(1-z)^{2b}}$ where $b \in \mathbb{C} \setminus \{0\}$, and $\alpha = \mu = p = 1, \beta = 0$ and $\gamma = \frac{1}{b}$ in Theorem 3.6, we have the following result obtained by the Srivatsava and Lashin [10].

Corollary 3.8. *Let $0 \neq b \in \mathbb{C}$. If $f \in \mathcal{A}$ and*

$$1 + \frac{1}{b} \left[\frac{zf'(z)}{f(z)} - 1 \right] \prec \frac{1+z}{1-z}$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}},$$

where $\frac{1}{(1-z)^{2b}}$ is the best dominant.

By taking $q(z) = (1+Bz)^{\frac{\lambda(A-B)}{B}}$, $\alpha = p = 1$, $\beta = 0$ and $\gamma = \frac{1}{\mu}$ in Theorem 3.6, then we have the following result of Obradović and Owa [7].

Corollary 3.9. *Let $-1 \leq B < A \leq 1$. Let μ, A and B satisfy the relation either $\left| \frac{\lambda(A-B)}{B} - 1 \right| \leq 1$ or $\left| \frac{\lambda(A-B)}{B} + 1 \right| \leq 1$. If $f \in \mathcal{A}$ and*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$$

then

$$\left(\frac{f(z)}{z} \right)^\mu \prec (1+Bz)^{\frac{\lambda(A-B)}{B}}$$

and $(1+Bz)^{\frac{\lambda(A-B)}{B}}$ is the best dominant.

Theorem 3.10. *Let α, β and γ be complex numbers and $\gamma \neq 0$. Let $q(z)$ be univalent in Δ with $q(0) = 1$. Let $f \in \mathcal{A}_p$ satisfy (3.1). Let*

$$(3.5) \quad \psi(z) := \frac{\alpha}{p} \left(\frac{z^p}{f(z)} \right)^\mu + \frac{\beta}{p^2} \left(\frac{z^p}{f(z)} \right)^{2\mu} + \gamma \mu \left[\left(\frac{z^p}{f(z)} \right)^\mu - \frac{1}{p} \frac{zf'(z)}{f(z)} \left(\frac{z^p}{f(z)} \right)^\mu \right].$$

If

$$\psi(z) \prec \alpha q(z) + \beta q^2(z) + \gamma z q'(z)$$

then

$$\frac{1}{p} \left(\frac{z^p}{f(z)} \right)^\mu \prec q(z)$$

and q is the best dominant.

Proof. The proof is a straight forward application of Theorem 2.1. □

By putting $\alpha = p = 1$, $\beta = 0$ and $\gamma = \frac{\lambda}{\mu}$ in Theorem 3.10 we have the following result:

Corollary 3.11. *If $f(z) \in \mathcal{A}$ and*

$$(1+\lambda) \left(\frac{z}{f(z)} \right)^\mu - \lambda f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} \prec q(z) + \frac{\lambda}{\mu} z q'(z)$$

then

$$\left(\frac{z}{f(z)} \right)^\mu \prec q(z).$$

By taking $\lambda = -1$ in Corollary 3.11 we get the following result.

Corollary 3.12. *If $f \in \mathcal{A}$ and*

$$f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} \prec q(z) - \frac{zq'(z)}{\mu}$$

implies

$$\left(\frac{z}{f(z)} \right)^{\mu} \prec q(z)$$

and q is the best dominant.

4. APPLICATION OF SUPERORDINATION

Theorem 4.1. *Let α, β and γ be complex numbers and $\gamma \neq 0$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfies*

$$(4.1) \quad \Re \left\{ \left(\frac{\alpha + 2\beta q(z)}{\gamma} \right) \right\} > 0.$$

Let

$$\psi(z) := \frac{\alpha}{p} \left(\frac{f(z)}{z^p} \right)^{\mu} + \frac{\beta}{p^2} \left(\frac{f(z)}{z^p} \right)^{2\mu} + \gamma \mu \left(\frac{f(z)}{z^p} \right)^{\mu} \left[\frac{zf'(z)}{f(z)} - 1 \right]$$

and is univalent in Δ . If $f \in \mathcal{A}_p$, $0 \neq \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^{\mu} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ then

$$\alpha q(z) + \beta q^2(z) + \gamma zq'(z) \prec \psi(z)$$

implies

$$q(z) \prec \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^{\mu}$$

where q is the best subdominant.

Proof. Define the function $\xi(z)$ by

$$(4.2) \quad \xi(z) := \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^{\mu}.$$

A computation using (4.2) shows that

$$\frac{z\xi'(z)}{\xi(z)} = \frac{z\mu f'(z)}{f(z)} - p\mu.$$

Note that

$$\begin{aligned} \psi(z) &:= \frac{\alpha}{p} \left(\frac{f(z)}{z^p} \right)^{\mu} + \frac{\beta}{p^2} \left(\frac{f(z)}{z^p} \right)^{2\mu} + \gamma \mu \left(\frac{f(z)}{z^p} \right)^{\mu} \left[\frac{zf'(z)}{f(z)} - 1 \right] \\ &= \alpha \xi(z) + \beta \xi^2(z) + \gamma z \xi'(z). \end{aligned}$$

Since $\alpha q(z) + \beta q^2(z) + \gamma zq'(z) \prec \psi(z)$, this can be written as (2.2), when $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \gamma$. Hence the result follows as an application of Theorem 2.2. $\square$

Corollary 4.2. Let $f \in \mathcal{A}$ and δ be complex number with $\Re \delta > 0$ and $-1 \leq B < A \leq 1$.

(i) If $(1 - \delta) \frac{f(z)}{z} + \delta z f'(z)$ is univalent in Δ , then

$$1 + Az \prec (1 - \delta) \frac{f(z)}{z} + \delta z f'(z) \Rightarrow 1 + \frac{A}{1 + \delta} z \prec \frac{f(z)}{z}$$

and $\frac{A}{1 + \delta} z$ is the best subdominant.

(ii) If $f'(z)$ is univalent in Δ then

$$\frac{1 + Az}{1 + Bz} \prec f'(z) \Rightarrow \frac{A}{B} + (1 - \frac{A}{B}) \frac{\log(1 + Bz)}{Bz} \prec \frac{f(z)}{z}$$

and $\frac{A}{B} + (1 - \frac{A}{B}) \frac{\log(1 + Bz)}{Bz}$ is the best subdominant.

Proof. Proof of first part follows from Theorem 4.1 by taking $\alpha = p = 1$, $\beta = 0$, $\gamma = \delta$, $\mu = 1$ and $q(z) := 1 + \frac{A}{1 + \delta} z$.

Proof of the second part follows from Theorem 4.1 by taking $\alpha = p = 1$, $\beta = 0$, $\gamma = \delta = 1$, $\mu = 1$ and $q(z) := \frac{A}{B} + (1 - \frac{A}{B}) \frac{\log(1 + Bz)}{Bz}$. $\square$

Theorem 4.3. Let α, β and γ be complex numbers and $\gamma \neq 0$. Let q be convex univalent in Δ and $\frac{z q'(z)}{q(z)}$ be starlike univalent in Δ . Further assume that

$$(4.3) \quad \Re \left\{ \frac{\beta q(z)}{\gamma} \right\} > 0.$$

Let $\alpha + \frac{\beta}{p} \left(\frac{f(z)}{z^p} \right)^\mu + \gamma \mu \left[\frac{z f'(z)}{f(z)} - p \right]$ is univalent in Δ . If $f \in \mathcal{A}_p$, $\frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ then

$$\alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \alpha + \frac{\beta}{p} \left(\frac{f(z)}{z^p} \right)^\mu + \gamma \mu \left[\frac{z f'(z)}{f(z)} - p \right]$$

implies

$$q(z) \prec \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu$$

where q is the best subdominant.

Proof. Let $\theta(w) := \alpha + \beta w$ and $\phi(w) := \frac{\gamma}{w}$. Note that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Hence the result follows as an application of Theorem 2.2, when $\xi(z) := \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu$. $\square$

Note that by taking $\alpha = p = 1$, $\beta = 0$, $\gamma = \frac{1}{\mu}$ and $q(z) := e^{\lambda A z}$ we get the corresponding superordination result of Corollary 3.7. Also by taking $\alpha = \mu = p = 1$, $\beta = 0$, $\gamma = \frac{1}{b}$ and $q(z) := \frac{1}{(1-z)^{2b}}$ we obtain the superordination result of Corollary 3.8

Theorem 4.4. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $\gamma \neq 0$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfy*

$$\Re \left\{ \left(\frac{\alpha + 2\beta q(z)}{\gamma} \right) \right\} > 0.$$

Let $\psi(z)$ as defined by (3.5) be univalent in Δ . If $f \in \mathcal{A}_p$ and $0 \neq \frac{1}{p} \left(\frac{z^p}{f(z)} \right)^\mu \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ then

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \psi(z)$$

implies

$$q(z) \prec \frac{1}{p} \left(\frac{z^p}{f(z)} \right)^\mu$$

where q is the best subdominant.

By letting $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{\lambda}{\mu}$ in Theorem 4.4, we get the following result:

Corollary 4.5. *Let $0 \neq \lambda \in \mathbb{C}$. Let q be convex univalent in Δ with $q(0) = 1$ and satisfy*

$$\Re \left\{ \frac{\mu}{\lambda} q'(z) \right\} > 0.$$

Let

$$\psi_1(z) := (1 + \lambda) \left(\frac{z}{f(z)} \right)^\mu - \lambda f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1}$$

be univalent in Δ . If $f \in \mathcal{A}$ and $\left(\frac{z}{f(z)} \right)^\mu \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ then

$$q(z) + \frac{\lambda}{\mu} q'(z) \prec \psi_1(z)$$

implies that

$$q(z) \prec \left(\frac{z}{f(z)} \right)^\mu$$

and q is the best subdominant.

By taking $\lambda = -1$ in Corollary 4.5 we get the following result:

Corollary 4.6. *If $f \in \mathcal{A}$ and*

$$q(z) - \frac{q'(z)}{\mu} \prec f'(z) \left(\frac{z}{f(z)} \right)^\mu$$

implies

$$q(z) \prec \left(\frac{z}{f(z)} \right)^\mu$$

and q is the best subdominant.

5. SANDWICH RESULTS

By combining Theorem 3.1 and Theorem 4.1 we get the following sandwich type result.

Theorem 5.1. *Let q_1 and q_2 be convex univalent in Δ , satisfying (4.1) and (3.1) respectively. Let $\psi(z)$ as given by (3.2) be univalent in Δ . If $0 \neq \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ then*

$$\alpha q_1(z) + \beta q_1^2(z) + \gamma z q_1'(z) \prec \psi(z) \prec \alpha q_2(z) + \beta q_2^2(z) + \gamma z q_2'(z)$$

implies

$$q_1(z) \prec \frac{1}{p} \left(\frac{f(z)}{z^p} \right)^\mu \prec q_2(z)$$

where q_1 and q_2 are respectively the best subdominant and best dominant.

Now by combining Theorem 3.6 and Theorem 4.3 with $p = 1$ we have the following result.

Theorem 5.2. *Let q_1 and q_2 be convex univalent in Δ , satisfying (4.3) and (3.4) respectively. Suppose $\frac{z q_i'(z)}{q_i(z)}$ be starlike univalent in Δ for $i = 1, 2$. Let*

$$\eta(z) := \alpha + \beta \left(\frac{f(z)}{z} \right)^\mu + \gamma \mu \left(\frac{z f'(z)}{f(z)} - 1 \right)$$

be univalent in Δ . If $0 \neq \left(\frac{f(z)}{z} \right)^\mu \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ then

$$\alpha + \beta q_1(z) + \gamma \frac{z q_1'(z)}{q_1(z)} \prec \eta(z) \prec \alpha + \beta q_2(z) + \gamma \frac{z q_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \left(\frac{f(z)}{z} \right)^\mu \prec q_2(z)$$

where q_1 and q_2 are respectively best subdominant and best dominant.

Theorem 5.3. *Let q_1 and q_2 be convex univalent satisfying (4.1) and (3.1) respectively. Let $0 \neq \left(\frac{f(z)}{z} \right)^\mu \in \mathcal{H}[1, 1] \cap \mathcal{Q}$.*

(i) Let $f \in \mathcal{A}$, and $(1 - \delta) \left(\frac{f(z)}{z} \right)^\mu + \delta f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1}$ is univalent in Δ then

$$\frac{1 + (1 - 2\beta_1)z}{1 - z} \prec (1 - \delta) \left(\frac{f(z)}{z} \right)^\mu + \delta f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \prec \frac{1 + (1 - 2\beta_2)z}{1 - z}$$

implies

$$1 + (-1 - \beta_1)(1 - \frac{2}{z} \log(1 - z)) \prec \left(\frac{f(z)}{z} \right)^\mu \prec 1 + (-1 - \beta_2)(1 - \frac{2}{z} \log(1 - z))$$

where $1 + (-1 - \beta_1)(1 - \frac{2}{z} \log(1 - z))$ and $1 + (-1 - \beta_2)(1 - \frac{2}{z} \log(1 - z))$ are respectively the best subdominant and best dominant.

(ii) If $(1 - \delta)\frac{f(z)}{z} + \delta f'(z)$ is univalent in Δ , then

$$1 + A_1 z \prec (1 - \delta)\frac{f(z)}{z} + \delta f'(z) \prec 1 + A_2 z$$

implies

$$1 + \frac{A_1}{1 + \delta} z \prec \frac{f(z)}{z} \prec 1 + \frac{A_2}{1 + \delta} z$$

where $1 + (\frac{A_1}{1 + \delta})z$ and $1 + (\frac{A_2}{1 + \delta})z$ are respectively the best subdominant and best dominant.

Proof. The proof of the first part follows from Theorem 5.1 by taking $q_i(z) = 1 + (1 - \beta_i)(-1 - 2 \log(1 - z))$ for $i = 1, 2$ and by taking $\alpha = p = 1, \beta = 0$ and $\gamma = \frac{\delta}{\mu}$ and the proof of second part follows by taking $q_i(z) = 1 + (\frac{A_i}{1 + \delta})z$ for $i = 1, 2$ and by taking $\alpha = \mu = p = 1, \beta = 0$ and $\gamma = \frac{\delta}{\mu}$. $\square$

In a similar manner we may obtain the sandwich result by combining Theorem 4.4 and Theorem 3.10.

6. APPLICATION TO KOMATU OPERATOR

Theorem 6.1. Let $h \in \mathcal{H}, h(0) = 1, h'(0) \neq 0$ and satisfy

$$\Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > -\frac{1}{2} \quad (z \in \Delta).$$

If $f \in \mathcal{A}_m$ satisfy the differential subordination

$$\frac{I_a^\sigma f(z)}{z} \prec h(z)$$

then

$$(6.1) \quad \frac{I_a^{\sigma+1} f(z)}{z} \prec g(z)$$

where

$$g(z) := \frac{1 + a}{mz^{\frac{1+a}{m}}} \int_0^z h(t) t^{\frac{1+a}{m}-1} dt.$$

The function g is convex and is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := \frac{I_a^{\sigma+1} f(z)}{z}.$$

A simple computation shows that

$$\frac{zp'(z)}{p(z)} = \left[\frac{z(I_a^{\sigma+1} f(z))'}{I_a^{\sigma+1} f(z)} - 1 \right].$$

By using the identity

$$z(I_a^{\sigma+1}f(z))' = (1+a)I_a^\sigma f(z) - aI_a^{\sigma+1}f(z),$$

we have

$$\frac{zp'(z)}{p(z)} = \left[\frac{(1+a)I_a^\sigma f(z)}{I_a^{\sigma+1}f(z)} - (a+1) \right],$$

and hence

$$p(z) + \frac{zp'(z)}{a+1} = \frac{I_a^\sigma f(z)}{z}.$$

The assertion (6.1) of Theorem 6.1 follows by an application of Theorem 2.3. $\square$

Theorem 6.2. *Let the function $q(z)$ be convex univalent in Δ and $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ and $q(z)$ satisfy*

$$(6.2) \quad \Re \left\{ \frac{\beta}{\delta} q(z) + \frac{2\gamma}{\delta} q^2(z) - \frac{zq'(z)}{q(z)} \right\} > 0$$

and let

$$(6.3) \quad \chi(z) := \alpha + \beta \left(\frac{I_a^{\sigma+1}f(z)}{z} \right)^\mu + \gamma \left(\frac{I_a^{\sigma+1}f(z)}{z} \right)^{2\mu} + \delta\mu(1+a) \left[\frac{I_a^\sigma f(z)}{I_a^{\sigma+1}f(z)} - 1 \right].$$

If

$$(6.4) \quad \chi(z) \prec \alpha + \beta q(z) + \gamma q^2(z) + \frac{\delta zq'(z)}{q(z)}$$

then

$$\left(\frac{I_a^{\sigma+1}f(z)}{z} \right)^\mu \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{I_a^{\sigma+1}f(z)}{z} \right)^\mu.$$

Note the function $p(z)$ is analytic in Δ . By a straight forward computation we have

$$(6.5) \quad \begin{aligned} \chi(z) &:= \alpha + \beta \left(\frac{I_a^{\sigma+1}f(z)}{z} \right)^\mu + \gamma \left(\frac{I_a^{\sigma+1}f(z)}{z} \right)^{2\mu} + \delta\mu(1+a) \left[\frac{I_a^\sigma f(z)}{I_a^{\sigma+1}f(z)} - 1 \right] \\ &= \alpha + \beta p(z) + \gamma p^2(z) + \frac{\delta zp'(z)}{p(z)}. \end{aligned}$$

By using (6.5) in subordination (6.4), we have

$$(6.6) \quad \alpha + \beta p(z) + \gamma p^2(z) + \frac{\delta z p'(z)}{p(z)} \prec \alpha + \beta q(z) + \gamma q^2(z) + \frac{\delta z q'(z)}{q(z)}.$$

The subordination (6.6) is same as (2.1) with $\theta(w) := \alpha + \beta w + \gamma w^2$ and $\phi(w) := \frac{\delta}{w}$. Clearly $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Hence the result follows as an application of Theorem 2.1. $\square$

Theorem 6.3. *Let $q(z)$ be convex univalent in Δ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ . Further assume that*

$$\Re \left\{ \frac{2\gamma}{\delta} q^2(z) + \frac{\beta}{\delta} q(z) \right\} > 0.$$

Let $\chi(z)$ as defined by (6.3), is univalent in Δ . If $f(z) \in \mathcal{A}$, $0 \neq \left(\frac{I_a^{\sigma+1} f(z)}{z} \right)^\mu \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ then

$$\alpha + \beta q(z) + \gamma q^2(z) + \frac{\delta z q'(z)}{q(z)} \prec \chi(z)$$

implies

$$q(z) \prec \left(\frac{I_a^{\sigma+1} f(z)}{z} \right)^\mu$$

and q is the best subordinant.

By combining Theorem 6.2 and Theorem 6.3 we obtain the sandwich result, however the details are omitted.

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