

SUBORDINATION RESULTS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

ZHI-GANG WANG AND R. AGHALARY

ABSTRACT. In the present paper, we derive some subordination results for certain classes of analytic functions by making use of a subordination theorem. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions of the form:

$$(1.1) \quad f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$.

For $0 \leq \alpha < 1$, we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the usual subclasses of $\mathcal{S}$ consisting of functions which are starlike of order α and convex of order α in $\mathbb{U}$, respectively, that is,

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha, z \in \mathbb{U} \right\},$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathbb{U} \right\}.$$

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.

Key words and phrases. Analytic functions; Hadamard product (or Convolution); Subordination between analytic functions; Salagean operator.

The present investigation was supported by the *Scientific Research Fund of Hunan Provincial Education Department* under Grant 08C118 of People's Republic of China.

Obviously, for any $0 \leq \alpha < 1$, we have

$$f \in \mathcal{K}(\alpha) \iff zf' \in \mathcal{S}^*(\alpha).$$

Let $\mathcal{T}(\lambda, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality:

$$\Re \left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right) > \alpha, \quad z \in \mathbb{U}$$

for some α ($0 \leq \alpha < 1$) and λ ($0 \leq \lambda \leq 1$), and let $\mathcal{C}(\lambda, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality:

$$\Re \left(z \frac{\lambda z^2 f'''(z) + (2\lambda + 1)zf''(z) + f'(z)}{\lambda z^2 f''(z) + z f'(z)} \right) > \alpha, \quad z \in \mathbb{U}$$

for some α ($0 \leq \alpha < 1$) and λ ($0 \leq \lambda \leq 1$). We note that

$$f \in \mathcal{C}(\lambda, \alpha) \iff zf' \in \mathcal{T}(\lambda, \alpha).$$

The classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ were introduced and investigated by Altıntaş [1], and Kamali and Akbulut [2], respectively.

Let $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality:

$$\Re \left(\frac{zf'(z)}{f(z)} \right) < \beta, \quad z \in \mathbb{U}$$

for some β ($\beta > 1$), and let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality:

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \beta, \quad z \in \mathbb{U}$$

for some β ($\beta > 1$). The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were introduced and investigated recently by Owa and Srivastava [5] (see also Nishiwaki and Owa [3], Owa and Nishiwaki [4], Srivastava and Attiya [7]).

Salagean [6] introduced the following operator:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N} := \{1, 2, \dots\}).$$

We note that

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Motivated by the above mentioned function classes, we now introduce the following subclasses of $\mathcal{A}$ involving the Salagean operator.

Definition 1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_n(\lambda, \alpha)$ if it satisfies the following inequality:

$$(1.2) \quad \Re \left(\frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^n f(z) + \lambda D^{n+1}f(z)} \right) > \alpha, \quad z \in \mathbb{U},$$

where

$$n \in \mathbb{N}_0, \quad 0 \leq \alpha < 1 \quad \text{and} \quad 0 \leq \lambda \leq 1.$$

It is easy to see that the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ are special cases of the class $\mathcal{S}_n(\lambda, \alpha)$.

Definition 2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_n(\lambda, \beta)$ if it satisfies the following inequality:

$$\Re \left(\frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^n f(z) + \lambda D^{n+1}f(z)} \right) < \beta, \quad z \in \mathbb{U},$$

where

$$n \in \mathbb{N}_0, \quad \beta > 1 \quad \text{and} \quad 0 \leq \lambda \leq 1.$$

It is also easy to see that the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ are special cases of the class $\mathcal{M}_n(\lambda, \beta)$.

We now provide some coefficient sufficient conditions for functions belonging to the classes $\mathcal{S}_n(\lambda, \alpha)$ and $\mathcal{M}_n(\lambda, \beta)$, which will be used in the proofs of our main theorems.

Lemma 1. Let $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ satisfies the following coefficient inequality:

$$(1.3) \quad \sum_{j=2}^{\infty} (j^{n+1} - \alpha j^n) (1 - \lambda + \lambda j) |a_j| \leq 1 - \alpha,$$

then $f \in \mathcal{S}_n(\lambda, \alpha)$.

Proof. To prove the claim, it suffices to show that

$$\left| \frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^n f(z) + \lambda D^{n+1}f(z)} - 1 \right| < 1 - \alpha, \quad z \in \mathbb{U}.$$

By noting that for any $z \in \mathbb{U}$, we have

$$\begin{aligned} & \left| \frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^n f(z) + \lambda D^{n+1}f(z)} - 1 \right| \\ &= \left| \frac{\sum_{j=2}^{\infty} [(1-\lambda)(j^{n+1} - j^n) + \lambda(j^{n+2} - j^{n+1})] a_j z^{j-1}}{1 + \sum_{j=2}^{\infty} [(1-\lambda)j^n + \lambda j^{n+1}] a_j z^{j-1}} \right| \\ &\leq \frac{\sum_{j=2}^{\infty} [(1-\lambda)(j^{n+1} - j^n) + \lambda(j^{n+2} - j^{n+1})] |a_j|}{1 - \sum_{j=2}^{\infty} [(1-\lambda)j^n + \lambda j^{n+1}] |a_j|}. \end{aligned}$$

It follows from (1.3) that the above last expression is bounded above by $1 - \alpha$. This completes the proof of Lemma 1. $\square$

Lemma 2. *Let $\beta > 1$ and $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$(1.4) \quad \sum_{j=2}^{\infty} [(1-\lambda)j^n + \lambda j^{n+1}] (j + |j - 2\beta|) |a_j| \leq 2(\beta - 1),$$

then $f \in \mathcal{M}_n(\lambda, \beta)$.

Proof. To prove $f \in \mathcal{M}_n(\lambda, \beta)$, it suffices to show that

$$(1.5) \quad \left| \frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^n f(z) + \lambda D^{n+1}f(z)} \right| < \left| \frac{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}{(1-\lambda)D^n f(z) + \lambda D^{n+1}f(z)} - 2\beta \right|.$$

We consider $M \in \mathbb{R}$ defined by

$$\begin{aligned} M &:= |(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)| \\ &\quad - |(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z) - 2\beta [(1-\lambda)D^n f(z) + \lambda D^{n+1}f(z)]| \\ &= \left| z + \sum_{j=2}^{\infty} [(1-\lambda)j^{n+1} + \lambda j^{n+2}] a_j z^j \right| \\ &\quad - \left| z + \sum_{j=2}^{\infty} [(1-\lambda)j^{n+1} + \lambda j^{n+2}] a_j z^j - 2\beta z - 2\beta \sum_{j=2}^{\infty} [(1-\lambda)j^n + \lambda j^{n+1}] a_j z^j \right|. \end{aligned}$$

Thus, for $|z| = r < 1$, we have

$$\begin{aligned} M &\leq r + \sum_{j=2}^{\infty} [(1-\lambda)j^{n+1} + \lambda j^{n+2}] |a_j| r^j \\ &\quad - \left[(2\beta - 1)r - \sum_{j=2}^{\infty} |[(1-\lambda)j^{n+1} + \lambda j^{n+2}] - 2\beta [(1-\lambda)j^n + \lambda j^{n+1}]| |a_j| r^j \right] \\ &< \left(\sum_{j=2}^{\infty} \{ [(1-\lambda)j^{n+1} + \lambda j^{n+2}] \right. \\ &\quad \left. + |[(1-\lambda)j^{n+1} + \lambda j^{n+2}] - 2\beta [(1-\lambda)j^n + \lambda j^{n+1}]| \} |a_j| - 2(\beta - 1) \right) r. \end{aligned}$$

It follows from (1.4) that $M < 0$, which implies that (1.5) holds true, hence $f \in \mathcal{M}_n(\lambda, \beta)$. $\square$

In view of Lemmas 1 and 2, we now introduce the following subclasses:

$$\widetilde{\mathcal{S}}_n(\lambda, \alpha) \subset \mathcal{S}_n(\lambda, \alpha) \quad \text{and} \quad \widetilde{\mathcal{M}}_n(\lambda, \beta) \subset \mathcal{M}_n(\lambda, \beta),$$

which consist of functions $f \in \mathcal{A}$ whose coefficients of the series satisfy the inequalities (1.3) and (1.4), respectively.

The main purpose of the present paper is to derive some subordination results for the classes $\widetilde{\mathcal{S}}_n(\lambda, \alpha)$ and $\widetilde{\mathcal{M}}_n(\lambda, \beta)$. To prove our main results, we also need the following definitions and lemma.

Definition 3 (Hadamard Product or Convolution). Given two functions $f, g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

Definition 4 (Subordination Principle). Given two functions f and g , analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function ω , which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1$$

such that

$$f(z) = g(\omega(z)).$$

It is easy to see that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \implies f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 5 (Subordination Factor Sequence). A sequence $\{b_j\}_{j=1}^{\infty}$ of complex numbers is said to be a subordination factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination

$$\sum_{j=1}^{\infty} a_j b_j z^j \prec f(z), \quad a_1 = 1, \quad z \in \mathbb{U}.$$

Lemma 3. (See Wilf [9]) *The sequence $\{b_j\}_{j=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re \left(1 + 2 \sum_{j=1}^{\infty} b_j z^j \right) > 0, \quad z \in \mathbb{U}.$$

2. SUBORDINATION RESULT FOR THE CLASS $\widetilde{\mathcal{S}}_n(\lambda, \alpha)$

We begin by presenting our first subordination result given by Theorem 6 below.

Theorem 6. *If $f \in \widetilde{\mathcal{S}}_n(\lambda, \alpha)$ and $g \in \mathcal{K}(0)$, then*

$$(2.1) \quad A_n(\lambda, \alpha) \cdot (f * g)(z) \prec g(z)$$

and

$$(2.2) \quad \Re(f) > -\frac{(1-\alpha) + 2^n(1+\lambda)(2-\alpha)}{2^n(1+\lambda)(2-\alpha)}$$

for any $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$, where, for convenience,

$$(2.3) \quad A_n(\lambda, \alpha) := \frac{2^{n-1}(1+\lambda)(2-\alpha)}{(1-\alpha) + 2^n(1+\lambda)(2-\alpha)}.$$

The constant factor $A_n(\lambda, \alpha)$ in the subordination result (2.1) is sharp, in the sense that $A_n(\lambda, \alpha)$ can not be replaced by a larger factor.

Proof. Let $f \in \widetilde{\mathcal{S}}_n(\lambda, \alpha)$ and suppose that

$$g(z) = z + \sum_{j=2}^{\infty} c_j z^j \in \mathcal{K} := \mathcal{K}(0).$$

Then

$$(2.4) \quad A_n(\lambda, \alpha) \cdot (f * g)(z) = A_n(\lambda, \alpha) \cdot \left(z + \sum_{j=2}^{\infty} a_j c_j z^j \right),$$

where $A_n(\lambda, \alpha)$ is defined by (2.3). Thus, by Definition 4, the subordination result (2.1) holds true if

$$\{A_n(\lambda, \alpha) \cdot a_j\}_{j=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. By Lemma 3, this is equivalent to the following inequality:

$$(2.5) \quad \Re \left(1 + \sum_{j=1}^{\infty} \frac{(1+\lambda)(2^{n+1} - \alpha 2^n)}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} a_j z^j \right) > 0, \quad z \in \mathbb{U}.$$

Since

$$(1 - \lambda + \lambda j)(j^{n+1} - \alpha j^n) \quad (j \geq 2; n \in \mathbb{N}_0)$$

is an increasing function of j , and using Lemma 1, we have

$$\begin{aligned}
& \Re \left(1 + \sum_{j=1}^{\infty} \frac{(1+\lambda)(2^{n+1} - \alpha 2^n)}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} a_j z^j \right) \\
&= \Re \left(1 + \frac{(1+\lambda)(2^{n+1} - \alpha 2^n)}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} a_1 z \right. \\
&\quad \left. + \frac{1}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} \cdot \sum_{j=2}^{\infty} (1+\lambda)(2^{n+1} - \alpha 2^n) a_j z^j \right) \\
&\geq 1 - \frac{(1+\lambda)(2^{n+1} - \alpha 2^n)}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} r \\
&\quad - \frac{1}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} \cdot \sum_{j=2}^{\infty} (1+\lambda)(2^{n+1} - \alpha 2^n) |a_j| r^j \\
&> 1 - \frac{(1+\lambda)(2^{n+1} - \alpha 2^n)}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} r - \frac{1-\alpha}{(1-\alpha) + (1+\lambda)(2^{n+1} - \alpha 2^n)} r \\
&= 1 - r > 0 \quad (|z| = r < 1).
\end{aligned}$$

This evidently proves the inequality (2.5), and hence also the subordination result (2.1) asserted by Theorem 6. The inequality (2.2) asserted by Theorem 6 follows from (2.1) by setting

$$g(z) = \frac{z}{1-z} = \sum_{j=1}^{\infty} z^j \in \mathcal{K}.$$

Finally, we consider the function f_0 defined by (2.6)

$$f_0(z) := z - \frac{1-\alpha}{(1+\lambda)(2^{n+1} - \alpha 2^n)} z^2 \quad (n \in \mathbb{N}_0; 0 \leq \lambda \leq 1; 0 \leq \alpha < 1),$$

which belongs to the class $\widetilde{\mathcal{S}}_n(\lambda, \alpha)$. Thus, by (2.1), we know that

$$A_n(\lambda, \alpha) \cdot f_0(z) \prec \frac{z}{1-z}, \quad z \in \mathbb{U}.$$

Furthermore, it can be easily verified for the function f_0 given by (2.6) that

$$\min_{z \in \mathbb{U}} \{ \Re (A_n(\lambda, \alpha) \cdot f_0(z)) \} = -\frac{1}{2}.$$

This complete the proof of Theorem 6. □

Remark 1. Setting $\lambda = 0$ in Theorem 6, we get the corresponding result obtained by Eker *et al.* [8].

3. SUBORDINATION RESULT FOR THE CLASS $\widetilde{\mathcal{M}}_n(\lambda, \alpha)$

The proof of the following subordination result is similar to that of Theorem 6. We, therefore, choose to omit the analogous details involved.

Theorem 7. *If $f \in \widetilde{\mathcal{M}}_n(\lambda, \alpha)$ and $g \in \mathcal{K}(0)$, then*

$$(3.1) \quad B_n(\lambda, \beta) \cdot (f * g)(z) \prec g(z)$$

and

$$\Re(f) > -\frac{\beta - 1 + 2^n\beta(1 + \lambda)}{2^n\beta(1 + \lambda)}$$

for any $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$, where, for convenience,

$$B_n(\lambda, \beta) := \frac{2^{n-1}\beta(1 + \lambda)}{\beta - 1 + 2^n\beta(1 + \lambda)}.$$

The constant factor $B_n(\lambda, \beta)$ in the subordination result (3.1) is sharp, in the sense that $B_n(\lambda, \beta)$ can not be replaced by a larger factor.

Remark 2. Putting $n = 0$ or 1 and $\lambda = 0$ in Theorem 7, we get the corresponding results obtained by Srivastava and Attiya [7].

4. ACKNOWLEDGEMENTS

The authors would like to thank the referee for his careful reading and making some valuable comments which have essentially improved the presentation of this paper.

REFERENCES

- [1] O. Altıntaş. On a subclass of certain starlike functions with negative coefficients. *Math. Japon.*, 36(3):489–495, 1991.
- [2] M. Kamali and S. Akbulut. On a subclass of certain convex functions with negative coefficients. *Appl. Math. Comput.*, 145(2-3):341–350, 2003.
- [3] J. Nishiwaki and S. Owa. Coefficient inequalities for certain analytic functions. *Int. J. Math. Math. Sci.*, 29(5):285–290, 2002.
- [4] S. Owa and J. Nishiwaki. Coefficient estimates for certain classes of analytic functions. *JIPAM. J. Inequal. Pure Appl. Math.*, 3(5):Article 72, 5 pp. (electronic), 2002.
- [5] S. Owa and H. M. Srivastava. Some generalized convolution properties associated with certain subclasses of analytic functions. *JIPAM. J. Inequal. Pure Appl. Math.*, 3(3):Article 42, 13 pp. (electronic), 2002.
- [6] G. Ş. Sălăgean. Subclasses of univalent functions. In *Complex analysis—fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981)*, volume 1013 of *Lecture Notes in Math.*, pages 362–372. Springer, Berlin, 1983.
- [7] H. M. Srivastava and A. A. Attiya. Some subordination results associated with certain subclasses of analytic functions. *JIPAM. J. Inequal. Pure Appl. Math.*, 5(4):Article 82, 6 pp. (electronic), 2004.
- [8] S. Sümer Eker, B. Şeker, and S. Owa. On subordination result associated with certain subclass of analytic functions involving Salagean operator. *J. Inequal. Appl.* Article ID 48294, 6pp., 2007.

- [9] H. S. Wilf. Subordinating factor sequences for convex maps of the unit circle. *Proc. Amer. Math. Soc.*, 12:689–693, 1961.

Received on June 20, 2007; accepted on February 1, 2009

SCHOOL OF MATHEMATICS AND COMPUTING SCIENCE,
CHANGSHA UNIVERSITY OF SCIENCE AND TECHNOLOGY,
YUNTANG CAMPUS,
CHANGSHA 410114, HUNAN, PEOPLE'S REPUBLIC OF CHINA
E-mail address: zhigwang@163.com

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF URMIA,
URMIA, IRAN
E-mail address: raghalary@yahoo.com