

INTEGRABILITY OF DISTRIBUTION $D^\perp$ ON A NEARLY SASAKIAN MANIFOLD

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ABSTRACT. In this paper, we give some sufficient and necessary conditions for integrability of distribution $D^\perp$ on a nearly Sasakian manifold, and generalize Bejancu's result.

1. INTRODUCTION

Let $\overline{M}$ be a real $(2n+1)$ -dimensional almost contact metric manifold with the structure tensors (Φ, ξ, η, g) , then

$$(1.1) \quad \Phi\xi = 0, \eta(\xi) = 1, \Phi^2 = -I + \eta \otimes \xi, \eta(X) = g(X, \xi),$$

$$(1.2) \quad g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta \circ \Phi = 0.$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.1 ([1]). The Nijenhuis tensor field of Φ on an almost contact metric manifold is defined by

$$(1.3) \quad [\Phi, \Phi](X, Y) = [\Phi X, \Phi Y] + \Phi^2[X, Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y],$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.2 ([1]). An almost contact metric manifold $\overline{M}$ is called a nearly Sasakian manifold, if we have

$$(1.4) \quad (\overline{\nabla}_X \Phi)Y + (\overline{\nabla}_Y \Phi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y,$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.3. An almost contact metric manifold $\overline{M}$ is called a Sasakian manifold, if we have

$$(1.5) \quad (\overline{\nabla}_X \Phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any $X, Y \in \Gamma(T\overline{M})$.

2000 *Mathematics Subject Classification.* 53C25.

Key words and phrases. nearly Sasakian manifold, semi-invariant submanifold, distribution, connection, integrable.

Supported by Foundation of Educational Committee of Hunan Province No. 05C267.

Obviously, a Sasakian manifold is a nearly Sasakian manifold.

Let M be an m -dimensional submanifold of an n -dimensional almost contact metric manifold $\overline{M}$. We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, denote by ∇ the induced connection on M , and denote by $\nabla^\perp$ the normal connection on M . Thus, for any $X, Y \in \Gamma(TM)$, we have

$$(1.6) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where $h: \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM^\perp)$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.6) is called the Gauss formula and h is called the second fundamental form of M .

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, we denote by $-A_V X$ and $\nabla_X^\perp V$ the tangent part and normal part of $\overline{\nabla}_X V$ respectively. Then we have

$$(1.7) \quad \overline{\nabla}_X V = -A_V X + \nabla_X^\perp V.$$

Thus, for any $V \in \Gamma(TM^\perp)$, we have a linear operator, satisfying

$$(1.8) \quad g(A_V X, Y) = g(X, A_V Y) = g(h(X, Y), V).$$

The equation (1.7) is called the Weingarten formula.

An m -dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assigns to each point x of $\overline{M}$ an m -dimensional linear subspace D_x of $T_x \overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x \in D_x$ for each $x \in \overline{M}$. When this happens we write $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $x \in \overline{M}$ there exist m differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of x . From now on, all distribution are supposed to be differentiable of class C^∞ .

The distribution D is said to be involutive if for all vector fields $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$. A submanifold M of $\overline{M}$ is said to be an integral manifold of D if for every point $x \in M$, D_x coincides with the tangent space to M at x . If there exists no integral manifold of D which contains M , then M is called a maximal integral manifold or a leaf of D . The distribution D is said to be integrable if for every $x \in \overline{M}$ there exists an integral manifold of D containing x .

Definition 1.4 ([1]). Let M be a real $(2m+1)$ -dimensional submanifold of a real $(2n+1)$ -dimensional almost contact metric manifold $\overline{M}$ with the structure tensors (Φ, ξ, η, g) . We assume that the structure tensor ξ is tangent to M , and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M . Then M is called a semi-invariant submanifold of $\overline{M}$, if there exist two differentiable distributions D and $D^\perp$ on M , satisfying

- (1) $TM = D \oplus D^\perp \oplus \{\xi\}$, where D , $D^\perp$ and $\{\xi\}$ are mutually orthogonal to each other;
- (2) the distribution D is invariant by Φ , that is, $\Phi(D_x) = D_x$, for each $x \in M$;
- (3) the distribution $D^\perp$ is anti-invariant by Φ , that is, $\Phi(D_x^\perp) \subset T_x M^\perp$, for each $x \in M$.

For each vector field X tangent to M , we put

$$(1.9) \quad \Phi X = \psi X + \omega X,$$

where ψX and ωX are respectively the tangent part and the normal part of ΦX . Also, for each vector field V normal to M , we put

$$(1.10) \quad \Phi V = BV + CV,$$

where BV and CV are respectively the tangent part and the normal part of ΦV .

In paper [1], we know that the distribution $D^\perp$ on M is integrable if and only if $[X, Y] \in \Gamma(D^\perp)$, for all vector fields $X, Y \in \Gamma(D^\perp)$.

2. MAIN RESULTS

Theorem 2.1. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$. Then the distribution $D^\perp$ is integrable if and only if*

$$(2.1) \quad g(A_{\Phi Y}X - A_{\Phi X}Y + 2(\bar{\nabla}_X \Phi)Y, \Phi Z) = \eta([X, Y])\eta(Z),$$

for any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Proof. By using (1.7) we obtain

$$(2.2) \quad \bar{\nabla}_X \Phi Y = -A_{\Phi Y}X + \nabla_X^\perp \Phi Y,$$

for any $X, Y \in \Gamma(D^\perp)$. On the other hand, by using (1.6) we also obtain

$$(2.3) \quad \bar{\nabla}_X \Phi Y = (\bar{\nabla}_X \Phi)Y + \Phi \bar{\nabla}_X Y = (\bar{\nabla}_X \Phi)Y + \Phi \nabla_X Y + \Phi h(X, Y).$$

By comparing (2.2) and (2.3) we get

$$(2.4) \quad (\bar{\nabla}_X \Phi)Y = -A_{\Phi Y}X + \nabla_X^\perp \Phi Y - \Phi \nabla_X Y - \Phi h(X, Y).$$

By changing X and Y in (2.4) we have

$$(2.5) \quad (\bar{\nabla}_Y \Phi)X = -A_{\Phi X}Y + \nabla_Y^\perp \Phi X - \Phi \nabla_Y X - \Phi h(Y, X).$$

By using (2.4) and (2.5) we obtain

$$(2.6) \quad (\bar{\nabla}_X \Phi)Y - (\bar{\nabla}_Y \Phi)X = -A_{\Phi Y}X + A_{\Phi X}Y + \nabla_X^\perp \Phi Y - \nabla_Y^\perp \Phi X - \Phi[X, Y].$$

By using (1.4)+(2.6) we get

$$(2.7) \quad 2(\bar{\nabla}_X \Phi)Y = 2g(X, Y)\xi - A_{\Phi Y}X + A_{\Phi X}Y + \nabla_X^\perp \Phi Y - \nabla_Y^\perp \Phi X - \Phi[X, Y].$$

That is,

$$(2.8) \quad \Phi[X, Y] = 2g(X, Y)\xi - A_{\Phi Y}X + A_{\Phi X}Y + \nabla_X^\perp \Phi Y - \nabla_Y^\perp \Phi X - 2(\bar{\nabla}_X \Phi)Y.$$

For any $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$. Hence, we have

$$(2.9) \quad g(2g(X, Y)\xi + \nabla_X^\perp \Phi Y - \nabla_Y^\perp \Phi X, \Phi Z) = 0.$$

By using (2.8), (1.2) and (2.9) we obtain

$$(2.10) \quad g([X, Y], Z) = g(-A_{\Phi Y}X + A_{\Phi X}Y - 2(\bar{\nabla}_X \Phi)Y, \Phi Z) + \eta([X, Y])\eta(Z).$$

Thus, $[X, Y] \in \Gamma(D^\perp)$ holds if and only if (2.1) is satisfied. $\square$

Lemma 2.1 ([1]). *Let M be a semi-invariant submanifold of a Sasakian manifold $\bar{M}$. Then*

$$(2.11) \quad A_{\Phi X}Y = A_{\Phi Y}X$$

and

$$(2.12) \quad [X, Y] \in \Gamma(D \oplus D^\perp),$$

for any $X, Y \in \Gamma(D^\perp)$.

Corollary (Bejancu-Papaghiuc [1]). *Let M be a semi-invariant submanifold of a Sasakian manifold $\bar{M}$. Then the distribution $D^\perp$ is integrable.*

Proof. For any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$ holds. By using Lemma 2.1 and (1.1) we obtain

$$(2.13) \quad g(-A_{\Phi Y}X + A_{\Phi X}Y, \Phi Z) = 0, \eta([X, Y]) = 0.$$

On the other hand, by using (1.5) we get

$$(2.14) \quad g(2(\bar{\nabla}_X \Phi)Y, \Phi Z) = 0.$$

Taking into account that a Sasakian manifold is a nearly Sasakian manifold, by using (2.13) and (2.14) we have (2.1). By using theorem 2.1, then the distribution $D^\perp$ is integrable. $\square$

Theorem 2.2. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$. Then the distribution $D^\perp$ is integrable if and only if*

$$(2.15) \quad g(A_{\Phi X}Y - A_{\Phi Y}X - 2\Phi \nabla_X Y, \Phi Z) = \eta([X, Y])\eta(Z),$$

for any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Proof. By using (2.4) and (2.8), we obtain

$$(2.16) \quad \Phi[X, Y] = 2g(X, Y)\xi + A_{\Phi Y}X - A_{\Phi X}Y + \nabla_X^\perp \Phi Y - \nabla_Y^\perp \Phi X + 2\Phi \nabla_X Y + 2\Phi h(X, Y),$$

for any $X, Y \in \Gamma(D^\perp)$. For any $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$. hence, we have

$$(2.17) \quad g(2g(X, Y)\xi + \nabla_X^\perp \Phi Y - \nabla_Y^\perp \Phi X, \Phi Z) = 0.$$

From (1.2), we get

$$(2.18) \quad g(2\Phi h(X, Y), \Phi Z) = 0.$$

By using (1.2), (2.16), (2.17) and (2.18), we obtain

$$(2.19) \quad g([X, Y], Z) = -g(A_{\Phi X}Y - A_{\Phi Y}X - 2\Phi \nabla_X Y, \Phi Z) + \eta([X, Y])\eta(Z).$$

Thus, $[X, Y] \in \Gamma(D^\perp)$ holds if and only if (2.15) is satisfied. $\square$

Lemma 2.2 ([4]). *Let $\bar{M}$ be a nearly Sasakian Manifold. Then*

$$(2.20) \quad \begin{aligned} [\Phi, \Phi](X, Y) = & 4\Phi(\bar{\nabla}_Y \Phi)X + \eta(Y)\bar{\nabla}_X \xi + X\eta(Y)\xi - \eta(\bar{\nabla}_X Y)\xi \\ & - 2g(X, \Phi Y)\xi + \eta(X)\Phi Y - \eta(X)\bar{\nabla}_Y \xi - Y\eta(X)\xi + \eta(\bar{\nabla}_Y X)\xi \\ & + 2g(Y, \Phi X)\xi - \eta(Y)\Phi X + 4g(X, Y)\xi - 2\eta(Y)X - 2\eta(X)Y, \end{aligned}$$

for any $X, Y \in \Gamma(T\bar{M})$.

Lemma 2.3. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$. Then*

$$(2.21) \quad \begin{aligned} 2\Phi[X, Y] = & -4g(X, Y)\xi - 2A_{\Phi Y}X + 2A_{\Phi X}Y + 2\nabla_X^\perp \Phi Y \\ & - 2\nabla_Y^\perp \Phi X - \Phi[\Phi, \Phi](X, Y) + 4\eta((\bar{\nabla}_Y \Phi)X)\xi, \end{aligned}$$

for any $X, Y \in \Gamma(D^\perp)$.

Proof. By using (2.20) and (1.1), we obtain

$$(2.22) \quad \Phi[\Phi, \Phi](X, Y) = 4\Phi^2(\bar{\nabla}_Y \Phi)X = -4(\bar{\nabla}_Y \Phi)X + 4\eta((\bar{\nabla}_Y \Phi)X)\xi,$$

for any $X, Y \in \Gamma(D^\perp)$. From (2.22) and (2.7), we get

$$(2.23) \quad \begin{aligned} \Phi[\Phi, \Phi](X, Y) = & -4g(X, Y)\xi - 2A_{\Phi Y}X + 2A_{\Phi X}Y + 2\nabla_X^\perp \Phi Y \\ & - 2\nabla_Y^\perp \Phi X - 2\Phi[X, Y] + 4\eta((\bar{\nabla}_Y \Phi)X)\xi. \end{aligned}$$

By using (2.23), we have (2.21). $\square$

Theorem 2.3. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$. Then the distribution $D^\perp$ is integrable if and only if*

$$(2.24) \quad g(2A_{\Phi Y}X - 2A_{\Phi X}Y + \Phi[\Phi, \Phi](X, Y), \Phi Z) = \eta([X, Y])\eta(Z),$$

for any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Proof. For any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$. Hence, we obtain

$$(2.25) \quad g(-4g(X, Y)\xi + 2\nabla_X^\perp \Phi Y - 2\nabla_Y^\perp \Phi X + 4\eta((\bar{\nabla}_Y \Phi)X)\xi, \Phi Z) = 0.$$

By using (2.21), (1.2) and (2.25), we get

$$(2.26) \quad \begin{aligned} 2g([X, Y], Z) = & g(2\Phi[X, Y], \Phi Z) + 2\eta([X, Y])\eta(Z) \\ = & g(-2A_{\Phi Y}X + 2A_{\Phi X}Y - \Phi[\Phi, \Phi](X, Y), \Phi Z) + \eta([X, Y])\eta(Z). \end{aligned}$$

Thus, $[X, Y] \in \Gamma(D^\perp)$ holds if and only if (2.24) is satisfied. $\square$

REFERENCES

- [1] A. Bejancu. *Geometry of CR-submanifolds*, volume 23 of *Mathematics and its Applications (East European Series)*. D. Reidel Publishing Co., Dordrecht, 1986.
- [2] H. Endō. Invariant submanifolds in a K -contact Riemannian manifold. *Tensor (N.S.)*, 28:154–156, 1974.
- [3] M. Okumura. Submanifolds of a Kaehlerian manifold and a Sasakian manifold. Lecture Notes, Michigan State Univ., 1971.
- [4] Y. Wan. Integrabilities of distribution $D \oplus \{\xi\}$ on a nearly Sasakian manifold. *J. Changsha Univ. Sci. Technol., Nat. Sci.*, 4(2):72–74, 2007.
- [5] K. Yano and S. Ishihara. Invariant submanifolds of an almost contact manifold. *Kōdai Math. Sem. Rep.*, 21:350–364, 1969.

Received on September 20, 2008., revised on May 12, 2009; accepted on June 28, 2009

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