

SUBRINGS IN TRIGONOMETRIC POLYNOMIAL RINGS

TARIQ SHAH AND EHSAN ULLAH

ABSTRACT. In this study we explore the subrings in trigonometric polynomial rings. Consider the rings T and T' of real and complex trigonometric polynomials over the fields $\mathbb{R}$ and its algebraic extension $\mathbb{C}$ respectively (see [6]). We construct the subrings T_0 of T and T'_0, T'_1 of T' . Then T_0 is a BFD whereas T'_0 and T'_1 are Euclidean domains. We also discuss among these rings the *Condition* : Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$.

1. INTRODUCTION

Following Cohn [3], an integral domain say D is atomic if each nonzero nonunit of D is a product of irreducible elements (*atoms*) of D , and it is well known that UFDs, PIDs and Noetherian domains are atomic domains. An integral domain D satisfies the *ascending chain condition on principal ideals (ACCP)* if there does not exist any infinite strictly ascending chain of principal integral ideals of D . Every PID, UFD and Noetherian domain satisfy ACCP and a domain satisfying ACCP is atomic. Grams [5] and Zaks [11] provided examples of atomic domains which do not satisfy ACCP. An integral domain D is a *bounded factorization domain (BFD)* if it is atomic and for each nonzero nonunit of D , there is a bound on the length of factorization into products of irreducible elements (cf. [1]). Examples of BFDs are UFDs and Noetherian or Krull domains (cf. [1, Proposition 2.2]). By [10], an integral domain D is said to be a *half-factorial domain (HFD)* if D is *atomic* and whenever $x_1 \dots x_m = y_1 \dots y_n$, where $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$ are irreducibles in D , then $m = n$. A UFD is obviously an HFD, but the converse fails, since any Krull domain D with $CI(D) \cong \mathbb{Z}_2$ is an HFD [10], but not a UFD. Moreover, a polynomial extension of an HFD is not an HFD, for example $\mathbb{Z}[\sqrt{-3}][X]$ is not an HFD, as $\mathbb{Z}[\sqrt{-3}]$ is an HFD but not integrally closed [4].

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In general,

$$\mathbf{UFD} \implies \mathbf{HFD} \implies \mathbf{BFD} \implies \mathbf{ACCP} \implies \mathbf{Atomic}.$$

But none of the above implications is reversible.

In integral domains, factorization properties have been a common interest of algebraists, particularly for polynomial rings. In this study, we would investigate the factorization properties of the subrings of trigonometric polynomial rings T and T' (see [6]). The basic concepts, notions and terminology are as standard in [6].

For the factorization of exponential polynomials, J. F. Ritt developed: “If $1 + a_1 e^{\alpha_1 x} + \dots + a_n e^{\alpha_n x}$ is divisible by $1 + b_1 e^{\beta_1 x} + \dots + b_r e^{\beta_r x}$ with no $b = 0$, then every β is a linear combination of $\alpha_1, \dots, \alpha_n$ with rational coefficients” [8, Theorem].

Latter on getting inspired by this, G. Picavet and M. Picavet [6] investigated some factorization properties in trigonometric polynomial rings. Following [6], when we replace all α_k above by im , with $m \in \mathbb{Z}$, we obtain trigonometric polynomials. Whereas

$$T' = \left\{ \sum_{k=0}^n (a_k \cos kx + b_k \sin kx) : n \in \mathbb{N}, a_k, b_k \in \mathbb{C} \right\} \text{ and}$$

$$T = \left\{ \sum_{k=0}^n (a_k \cos kx + b_k \sin kx) : n \in \mathbb{N}, a_k, b_k \in \mathbb{R} \right\}$$

are the trigonometric polynomial rings.

Again following [6], $\sin^2 x = (1 - \cos x)(1 + \cos x)$ shows that two different non-associated irreducible factorizations of the same element may appear. Throughout we denote by $\cos kx$ and $\sin kx$ the two functions $x \mapsto \cos kx$ and $x \mapsto \sin kx$ (defined over $\mathbb{R}$). Also from basic trigonometric identities, it is obvious that for each $n \in \mathbb{N} \setminus \{1\}$, $\cos nx$ represents a polynomial in $\cos x$ with degree n and $\sin nx$ represents the product of $\sin x$ and a polynomial in $\cos x$ with degree $n - 1$. Conversely by linearization formulas, it follows that any product $\cos^n x \sin^p x$ can be written as:

$$\sum_{k=0}^q (a_k \cos kx + b_k \sin kx), \text{ where } q \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{Q}.$$

Hence $T = \mathbb{R}[\cos x, \sin x] \subseteq \mathbb{C}[\cos x, \sin x] = T'$.

Here T' is a Euclidean domain and T is a Dedekind half-factorial domain (see [6, Theorem 2.1 & Theorem 3.1]). We continue the investigations to find the factorization properties in trigonometric polynomial rings, begun in [6]. In other words we extend this study towards finding factorization properties of the subrings of trigonometric polynomial rings, by establishing T_0 , T'_0 , and T'_1 as subrings.

In this paper we explored T_0 , T'_0 and T'_1 and demonstrated that, the ring T'_0 and T'_1 are Euclidean domains, whereas T_0 is a BFD. We also characterized the irreducible elements of T'_0 , and discussed *Condition 1* (see [7, page 661]) among trigonometric polynomial rings.

2. THE SUBRINGS OF $\mathbb{C}[\cos x, \sin x]$

The Construction of T'_1 . We consider

$$T'_1 = \left\{ \sum_{k=0}^n (a_k \cos kx + ib_k \sin kx), \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{R} \right\}.$$

Let

$$z = \sum_{k=0}^n (a_k \cos kx + ib_k \sin kx) \in T'_1,$$

As $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, so

$$z = e^{-inx} \left[\sum_{k=0}^n \left\{ \left(\frac{a_k + b_k}{2} \right) e^{i(n+k)x} + \left(\frac{a_k - b_k}{2} \right) e^{i(n-k)x} \right\} \right],$$

where $\frac{a_k + b_k}{2}, \frac{a_k - b_k}{2} \in \mathbb{R}$. Since z is an arbitrary, therefore every element of T'_1 is of the form

$$e^{-inx} P(e^{ix}), \ n \in \mathbb{N}, \text{ where } P(X) \in \mathbb{R}[X].$$

Conversely,

$$e^{-inx} P(e^{ix}) = \sum_{k=0}^{n-1} (\alpha_k e^{-i(n-k)x} + \alpha_{2n-k} e^{i(n-k)x}) + \alpha_n,$$

where $\alpha_k \in \mathbb{R}$. As $e^{ix} = \cos x + i \sin x$, so

$$\begin{aligned} e^{-inx} P(e^{ix}) &= \sum_{k=0}^{n-1} \{ (\alpha_k + \alpha_{2n-k}) \cos(n-k)x + \\ &\quad i(\alpha_{2n-k} - \alpha_k) \sin(n-k)x \} + \alpha_n, \end{aligned}$$

where $\alpha_k + \alpha_{2n-k}, \alpha_{2n-k} - \alpha_k \in \mathbb{R}$. Therefore every element which is of the form $e^{-inx} P(e^{ix}), n \in \mathbb{N}$, where $P(X) \in \mathbb{R}[X]$, is in T'_1 .

Conclusion 1. The consequence of above construction is:

$$T'_1 = \{ e^{-inx} P(e^{ix}), \ n \in \mathbb{N}, \text{ where } P(X) \in \mathbb{R}[X] \}.$$

So we have an isomorphism $f: (\mathbb{R}[X])_X \rightarrow T'_1$ through the substitution morphism $X \rightarrow e^{ix}$. Therefore $T'_1 \simeq (\mathbb{R}[X])_X$.

Theorem 1. T'_1 is a Euclidean domain.

Proof. $(\mathbb{R}[X])_X$ is a localization of $\mathbb{R}[X]$ by a multiplicative system generated by a prime X . Also $\mathbb{R}[X]$ is a Euclidean domain. Therefore $(\mathbb{R}[X])_X$ is a Euclidean domain [9, Proposition 7]. Hence the isomorphism $T'_1 \simeq (\mathbb{R}[X])_X$ in Conclusion 1 gives the result. $\square$

The Construction of T'_0 . We define the set T'_0 of all polynomials of the form

$$\sum_{k=0}^n (a_k \cos kx + b_k \sin kx),$$

$n \in \mathbb{N}$, $a_k, b_k \in \mathbb{C}$ and $a_n = \alpha + \gamma + i\beta$, $b_n = -\beta + i(\alpha - \gamma)$ such that $\alpha, \beta, \gamma \in \mathbb{R}$, α, β and γ are not simultaneously zero. Let $z \in T'_0$ be an arbitrary element, so we may write

$$z = a_0 + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + \{(\alpha + \gamma + i\beta) \cos nx + (-\beta + i(\alpha - \gamma)) \sin nx\},$$

As $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, so

$$z = a_0 + \sum_{k=1}^{n-1} \left\{ \left(\frac{a'_k + b''_k + i(a''_k - b'_k)}{2} \right) e^{ikx} + \left(\frac{a'_k - b''_k + i(a''_k + b'_k)}{2} \right) e^{-ikx} \right\} + (\alpha + i\beta)e^{inx} + \gamma e^{-inx},$$

where $a_k = a'_k + ia''_k$, $b_k = b'_k + ib''_k$ and $a'_k, a''_k, b'_k, b''_k \in \mathbb{R}$, $a_0 \in \mathbb{C}$. Setting

$$\alpha'_k = \frac{a'_k + b''_k + i(a''_k - b'_k)}{2} \text{ and } \beta'_k = \frac{a'_k - b''_k + i(a''_k + b'_k)}{2},$$

we have

$$z = e^{-inx} \left[a_0 e^{inx} + \sum_{k=1}^{n-1} \{ \alpha'_k e^{i(n+k)x} + \beta'_k e^{i(n-k)x} \} + (\alpha + i\beta) e^{i2nx} + \gamma \right],$$

where $\alpha'_k, \beta'_k, a_0 \in \mathbb{C}$, and $\alpha, \beta, \gamma \in \mathbb{R}$. Since z is an arbitrary, therefore every element of T'_0 is of the form

$$e^{-inx} P(e^{ix}), \quad n \in \mathbb{N}, \text{ where } P(X) \in \mathbb{R} + X\mathbb{C}[X].$$

Conversely,

$$e^{-inx} P(e^{ix}) = \alpha_0 e^{-inx} + \alpha_{2n} e^{inx} + \sum_{k=1}^{n-1} (\alpha_k e^{-i(n-k)x} + \alpha_{2n-k} e^{i(n-k)x}) + \alpha_n,$$

where $\alpha_0 \in \mathbb{R}$, $\alpha_k \in \mathbb{C}$. Let

$$\alpha_k = \alpha'_k + i\alpha''_k, \alpha_{2n-k} = \alpha'_{2n-k} + i\alpha''_{2n-k}, \alpha_{2n} = \alpha'_{2n} + i\alpha''_{2n}.$$

So for $e^{ix} = \cos x + i \sin x$, we have

$$\begin{aligned}
e^{-inx} P(e^{ix}) &= (\alpha_0 + \alpha'_{2n} + i\alpha''_{2n}) \cos nx + (-\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0)) \sin nx \\
&\quad + \sum_{k=1}^{n-1} \{(\alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k})) \cos(n-k)x \\
&\quad + (\alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k)) \sin(n-k)x\} + \alpha_n \\
&= a_n \cos nx + b_n \sin nx \\
&\quad + \sum_{k=1}^{n-1} \{a_k \cos(n-k)x + b_k \sin(n-k)x\} + \alpha_n,
\end{aligned}$$

where

$$\begin{aligned}
a_n &= \alpha_0 + \alpha'_{2n} + i\alpha''_{2n}, \quad b_n = -\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0), \\
a_k &= \alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}) \\
b_k &= \alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k).
\end{aligned}$$

Therefore every element which is of the form $e^{-inx} P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{R} + X\mathbb{C}[X]$, is in T'_0 .

Conclusion 2. The consequence of above construction is:

$$T'_0 = \{e^{-inx} P(e^{ix}), n \in \mathbb{N}, \text{ where } P(X) \in \mathbb{R} + X\mathbb{C}[X]\}.$$

So again we have an isomorphism $f: (\mathbb{R} + X\mathbb{C}[X])_X \rightarrow T'_0$ through the substitution morphism $X \rightarrow e^{ix}$. Therefore $T'_0 \simeq (\mathbb{R} + X\mathbb{C}[X])_X$.

Theorem 2. *The integral domain T'_0 is a Euclidean domain having irreducible elements, up to units, trigonometric polynomials of the form $\cos x + i \sin x - a$, where $a \in \mathbb{C} \setminus \{0\}$.*

Proof. Since $(\mathbb{R} + X\mathbb{C}[X])_X = \mathbb{C}[X, 1/X] = \mathbb{C}[X]_X$ is a UFD (PID, Euclidean domain, etc.). Thus the domain $(\mathbb{R} + X\mathbb{C}[X])_X$ is a Euclidean domain. Now use the isomorphism $T'_0 \simeq (\mathbb{R} + X\mathbb{C}[X])_X$ in Conclusion 2. $\square$

The following assertion is the analogue of [6, Corollary 2.2] and gives the factorization in T'_0 instead of T' .

Corollary 1. *Let $z = \sum_{k=0}^n (a_k \cos kx + b_k \sin kx)$, $n \in \mathbb{N} \setminus \{1\}$, $a_k, b_k \in \mathbb{C}$ with $(a_n, b_n) \neq (0, 0)$, such that $a_n = \alpha + \gamma + i\beta$ and $b_n = -\beta + i(\alpha - \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Let d be a common divisor of the integers k such that $(a_k, b_k) \neq (0, 0)$. Then z has a unique factorization*

$$\lambda(\cos nx - i \sin nx) \prod_{j=1}^{\frac{2n}{d}} (\cos dx + i \sin dx - \alpha_j), \text{ where } \lambda, \alpha_j \in \mathbb{C} \setminus \{0\}.$$

Proof. Since $T'_0 \subseteq T'$, therefore proof follows by [6, Corollary 2.2]. $\square$

Now onwards the symbol $\cap$ in all diagrams will represent the inclusion $\subseteq$.

Remark 1. $\mathbb{R}+X\mathbb{C}[X]$ is a Noetherian HFD wedged between two Euclidean domains $\mathbb{R}[X]$ and $\mathbb{C}[X]$, that is $\mathbb{R}[X] \subseteq \mathbb{R}+X\mathbb{C}[X] \subseteq \mathbb{C}[X]$ and the localization of all these by a multiplicative system generated by X preserves their factorization properties in the following way

$$\begin{array}{ccccc} \mathbb{R}[X] & \subseteq & \mathbb{R}+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & & \cap & & \cap \\ (\mathbb{R}[X])_X & \subseteq & (\mathbb{R}+X\mathbb{C}[X])_X & \subseteq & (\mathbb{C}[X])_X. \end{array}$$

Using Conclusion 1, Conclusion 2 and [6, Theorem 2.1], we have

$$\begin{array}{ccccc} \mathbb{R}[X] & \subseteq & \mathbb{R}+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & & \cap & & \cap \\ T'_1 & \subseteq & T'_0 & \subseteq & T', \end{array}$$

where T'_0 is a Euclidean domain wedged between two Euclidean domains T'_1 and T' .

Remark 2. (a) Consider the domain extension $\mathbb{R}[X] \subseteq (\mathbb{R}[X])_X$. As $X\mathbb{R}[X]$ is a maximal ideal of $\mathbb{R}[X]$ and $X\mathbb{R}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{R}[X])^e = (\mathbb{R}[X])_X$ [12, Corollary 2]. Hence $(X\mathbb{R}[X])^e \simeq T'_1$ by Conclusion 1.

(b) If we consider the domain extension $\mathbb{R}+X\mathbb{C}[X] \subseteq (\mathbb{R}+X\mathbb{C}[X])_X$. We observe, that $X\mathbb{C}[X]$ is a maximal ideal of $\mathbb{R}+X\mathbb{C}[X]$ and $X\mathbb{C}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{C}[X])^e = (\mathbb{R}+X\mathbb{C}[X])_X$ [12, Corollary 2]. Hence $(X\mathbb{C}[X])^e \simeq T'_0$ by Conclusion 2.

(c) On the same lines we can apply the same result to the domain extension $\mathbb{C}[X] \subseteq (\mathbb{C}[X])_X$. In this case $X\mathbb{C}[X]$ is a maximal ideal of $\mathbb{C}[X]$ and $X\mathbb{C}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{C}[X])^e = (\mathbb{C}[X])_X$ [12, Corollary 2]. Hence $(X\mathbb{C}[X])^e \simeq T'$ by [6, Theorem 2.1].

Definition 1. Let J be a subset of T'_1 defined by

$$J = \left\{ \sum_{k=0}^n (a_k \cos kx + ib_k \sin kx), n \in \mathbb{N}, a_k, b_k \in \mathbb{Q} \text{ and } a_n = b_n \right\}.$$

Definition 2. Let I be a subset of T'_0 defined by

$$I = \left\{ \sum_{k=0}^n (a_k \cos kx + b_k \sin kx) : n \in \mathbb{N}, a_k, b_k \in \mathbb{C} \right. \\ \left. \text{and } a_n = \alpha + i\beta, b_n = -\beta + i\alpha \right\}.$$

Lemma 1. For the maximal ideal $X\mathbb{R}[X]$ (respectively $X\mathbb{C}[X]$) of $\mathbb{R}[X]$ (respectively $\mathbb{R}+X\mathbb{C}[X]$), we have $(X\mathbb{R}[X])_X \simeq J$ (respectively $(X\mathbb{C}[X])_X \simeq I$).

Proof. Follows by Conclusion 1 (respectively Conclusion 2). $\square$

Condition 1. Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$ [7, page 661].

Example 1. (a) If the ring extension $A \subseteq B$ satisfies Condition 1, then the ring extension $A + XB[X] \subseteq B[X]$ (or $A + XB[[X]] \subseteq B[[X]]$) also satisfies Condition 1.

(b) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 1, then so does the ring extension $A \subseteq C$.

(c) If B is a fraction ring of A , then the ring extension $A \subseteq B$ satisfies Condition 1. Hence the ring extension $A \subseteq B$ satisfies Condition 1 is the generalization of localization.

(d) If B is a field, then the ring extension $A \subseteq B$ satisfies Condition 1.

Condition 2. Let A, A_1, B, B_1 be unitary (commutative) rings such that

$$\begin{array}{ccc} A & \subseteq & B \\ \cap & & \cap \\ A_1 & \subseteq & B_1 \end{array}$$

Then for each $x \in B_1$ there exist $x' \in U(B)$ and $x'' \in A_1$ such that $x = x'x''$.

Lemma 2. Let $A \subseteq B$ be a unitary (commutative) ring extension which satisfies Condition 1. If N is a multiplicative system in A then the ring extension $N^{-1}A \subseteq N^{-1}B$ satisfies Condition 2.

Proof. Since the ring extension $A \subseteq B$ satisfies Condition 1. Therefore for each $a \in B$ there exist $b \in U(B)$ and $c \in A$ such that $a = bc$. Obviously $N^{-1}A \subseteq N^{-1}B$ and let $x = \frac{a}{s} \in N^{-1}B$. Then $x = \frac{a}{s}$, $a \in B$, $s \in N$. This implies $x = \frac{bc}{s} = b\frac{c}{s}$, where $b \in U(B)$ and $\frac{c}{s} \in N^{-1}A$. $\square$

Example 2. (a) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 2, then so does the ring extension $A \subseteq C$.

(b) For $A = A_1$ and $B = B_1$ the Condition 1 and Condition 2 coincides.

(c) If the ring extension $A_1 \subseteq B_1$ satisfies Condition 2, then it does satisfies Condition 1.

(d) By Lemma 2, the ring extensions $T'_1 \subseteq T'_0$ and $T'_0 \subseteq T'$ satisfy Condition 2 so does the ring extension $T'_1 \subseteq T'$.

Remark 3. Consider the commutative inclusion diagram made by the following domain extensions

$$\begin{array}{ccccc} \mathbb{R}[X] & \subseteq & \mathbb{R}+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & \searrow & \cap & \searrow & \cap \\ T'_1 & \subseteq & T'_0 & \subseteq & T'. \end{array}$$

Among these domain extensions $\mathbb{R}+X\mathbb{C}[X] \subseteq \mathbb{C}[X]$, $\mathbb{R}[X] \subseteq T'_1$, $\mathbb{R}+X\mathbb{C}[X] \subseteq T'_0$ and $\mathbb{C}[X] \subseteq T'$ satisfy Condition 1 (see Example 1). Whereas the domain extensions $T'_0 \subseteq T'$ and $T'_1 \subseteq T'$ satisfy Condition 2. So by transitivity the domain extensions $\mathbb{R}[X] \subseteq T'_0$, $\mathbb{R}+X\mathbb{C}[X] \subseteq T'$ and $T'_1 \subseteq T'$ also satisfy Condition 2. Also note that the domain extension $\mathbb{R}[X] \subseteq \mathbb{R}+X\mathbb{C}[X]$ does not satisfy any of Condition 1 and Condition 2.

The subring of $\mathbb{R}[\cos x, \sin x]$. Consider the substitution morphism

$$g: \mathbb{Z}[X, Y] \rightarrow \mathbb{Z}[\cos x, \sin x],$$

defined by $g(X) = \cos x$ and $g(Y) = \sin x$ such that

$$g(X^2 + Y^2 - 1) = g(X^2) + g(Y^2) - 1 = \cos^2 x + \sin^2 x - 1 = 0.$$

This implies $(X^2 + Y^2 - 1) = \text{Ker } g$, therefore

$$\mathbb{Z}[\cos x, \sin x] \simeq \mathbb{Z}[X, Y]/(X^2 + Y^2 - 1).$$

Theorem 3. *The integral domain $T_0 = \mathbb{Z}[\cos x, \sin x]$ is a BFD.*

Proof. Since $\mathbb{Z}[X, Y]/(X^2 + Y^2 - 1) \simeq \mathbb{Z}[\cos x, \sin x]$, with $\mathbb{Z}[X, Y]$ a Noetherian domain. Therefore $\mathbb{Z}[\cos x, \sin x]$ is Noetherian, hence the result. $\square$

Remark 4. (a) T is a Dedekind HFD [6, Theorem 3.1], whereas T_0 is a Noetherian BFD.

- (b) T_0 is a free $\mathbb{Z}[\cos x]$ -module and has basis $\{1, \sin x\}$.
- (c) $\mathbb{Z}[\cos x]$ is a Euclidean domain because $\mathbb{Z}[\cos x] \simeq \mathbb{Z}[X]$, therefore the BFD T_0 lies between Euclidean domains $\mathbb{Z}[\cos x]$ and T'_0 .
- (d) T'_0 is a free T_0 -module and has basis $\{1, i\}$.
- (e) T' is a T_0 -module also T is a T_0 -module.
- (f) T' is a T'_0 -module.

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QUAID-I-AZAM UNIVERSITY, PAKISTAN

E-mail address: `stariqshah@gmail.com`

PASSAU UNIVERSITÄT, GERMANY

E-mail address: `ehsanu01@stud.uni-passau.de`