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CERTAIN SUBCLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS DEFINED BY CONVOLUTION

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ABSTRACT. The aim of this paper is to obtain coefficient estimates, distortion theorems, convex linear combinations and radii of close-to-convexity, starlikeness and convexity for functions belonging to the subclass $TS_\gamma(f, g; \alpha, \beta)$ of uniformly starlike and convex functions, we consider integral operators associated with functions in this class. Furthermore partial sums $f_n(z)$ of functions $f(z)$ in the class $TS_\gamma(f, g; \alpha, \beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f^{'}(z)$ to $f_n^{'}(z)$ are determined.

1. INTRODUCTION

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

that are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. Let $f \in S$ be given by (1.1) and $g \in S$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0), \tag{1.2}$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Following Goodman ([4] and [5]), Ronning ([9] and [10]) introduced and studied the following subclasses:

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(i) A function $f(z)$ of the form (1.1) is said to be in the class $S_p(\alpha,\beta)$ of uniformly $\beta-$starlike functions if it satisfies the condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right| \quad (z\in U), \tag{1.4}$$

where $-1\le\alpha<1$ and $\beta\ge 0$.

(ii) A function $f(z)$ of the form (1.1) is said to be in the class $UCV(\alpha,\beta)$ of uniformly $\beta-$convex functions if it satisfies the condition:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (z\in U), \tag{1.5}$$

where $-1\le\alpha<1$ and $\beta\ge 0$.

It follows from (1.4) and (1.5) that

$$f(z)\in UCV(\alpha,\beta)\Longleftrightarrow zf'(z)\in S_p(\alpha,\beta). \tag{1.6}$$

For $-1\le\alpha<1$, $0\le\gamma\le 1$ and $\beta\ge 0$, we let $S_\gamma(f,g;\alpha,\beta)$ be the subclass of S consisting of functions $f(z)$ of the form (1.1) and the functions $g(z)$ of the form (1.2) and satisfying the analytic criterion:

$$\begin{aligned}\operatorname{Re}&\left\{\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-\alpha\right\}\\ &>\beta\left|\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right|.\end{aligned} \tag{1.7}$$

Let T denote the subclass of S consisting of functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k \quad (a_k\ge 0). \tag{1.8}$$

Further, we define the class $TS_\gamma(f,g;\alpha,\beta)$ by

$$TS_\gamma(f,g;\alpha,\beta)=S_\gamma(f,g;\alpha,\beta)\cap T. \tag{1.9}$$

We note that:

(i) $TS_0(f,\frac{z}{(1-z)};\alpha,1)=S_pT(\alpha)$ and $TS_0(f,\frac{z}{(1-z)^2};\alpha,1)=TS_1(f,\frac{z}{(1-z)};\alpha,1)=UCT(\alpha)$, $(-1\le\alpha<1)$ (see Bharati et al. [3]);

(ii) $TS_1(f,\frac{z}{(1-z)};0,\beta)=UCT(\beta)$ $(\beta\ge 0)$ (see Subramanian et al. [15]);

(iii) $TS_0(f,z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k;\alpha,\beta)=TS(\alpha,\beta)$ $(-1\le\alpha<1,\ \beta\ge 0,\ c\ne 0,-1,-2,\ldots)$ (see Murugusundaramoorthy and Magesh [6,7]);

(iv) $TS_0(f,z+\sum_{k=2}^{\infty}k^nz^k;\alpha,\beta)=TS(n,\alpha,\beta)$ $(-1\le\alpha<1,\ \beta\ge 0,\ n\in N_0=N\cup\{0\},\ N=\{1,2,\ldots\})$(see Rosy and Murugusundaramoorthy [11]);

(v) $TS_0(f, z + \sum_{k=2}^{\infty} \binom{k+\lambda-1}{\lambda} z^k; \alpha, \beta) = D(\beta, \alpha, \lambda)\,(-1 \le \alpha < 1, \beta \ge 0,$ $\lambda > -1)$ (see Shams et al. [14]);

(vi) $TS_0(f, z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n z^k; \alpha, \beta) = TS_\lambda(n, \alpha, \beta)$ $(-1 \le \alpha < 1,$ $\beta \ge 0$, $\lambda \ge 0$, $n \in N_0)$ (see Aouf and Mostafa [2]);

(vii) $TS_\gamma(f, z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta) = TS(\gamma, \alpha, \beta)\,(-1 \le \alpha < 1,$ $\beta \ge 0,$ $0 \le \gamma \le 1$, $c \ne 0, -1, -2, \ldots)$ (see Murugusundaramoorthy et al. [8]);

(viii) $TS_\gamma(f, z + \sum_{k=2}^{\infty} \Gamma_k z^k; \alpha, \beta) = TS_q^s(\gamma, \alpha, \beta)$ (see Ahuja et al. [1]), where

$$\Gamma_k = \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1 \ldots}(\beta_s)_{k-1}} \frac{1}{(k-1)!} \tag{1.10}$$

$(\alpha_i > 0,\ i = 1, \ldots, q;\ \beta_j > 0,\ j = 1, \ldots, s;\ q \le s + 1;\ q,\ s \in N_0)$.

Also we note that

$$\begin{aligned} TS_\gamma(f, z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta) &= TS_\gamma(n, \alpha, \beta) \\ = \Bigg\{ f \in T : \operatorname{Re}\left\{ \frac{(1-\gamma)z(D^n f(z))' + \gamma z (D^{n+1} f(z))'}{(1-\gamma)D^n f(z) + \gamma D^{n+1} f(z)} - \alpha \right\} \\ > \beta \left| \frac{(1-\gamma)z(D^n f(z))' + \gamma z (D^{n+1} f(z))'}{(1-\gamma)D^n f(z) + \gamma D^{n+1} f(z)} - 1 \right|, \\ -1 \le \alpha < 1,\ \beta \ge 0,\ n \in N_0,\ z \in U \Bigg\}. \end{aligned} \tag{1.11}$$

2. Coefficient estimates

Theorem 1. *A function $f(z)$ of the form (1.8) is in $TS_\gamma(f, g; \alpha, \beta)$ if*

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)]\,[1 + \gamma(k-1)]\,|a_k|\, b_k \le 1 - \alpha, \tag{2.1}$$

where $-1 \le \alpha < 1$, $\beta \ge 0$ *and* $0 \le \gamma \le 1$.

Proof. It suffices to show that

$$\beta \left| \frac{z(f*g)'(z) + \gamma z^2 (f*g)''(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)'(z)} - 1 \right| - \operatorname{Re}\left\{ \frac{z(f*g)'(z) + \gamma z^2 (f*g)''(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)'(z)} - 1 \right\} \le 1 - \alpha.$$

We have

$$\begin{aligned}
&\beta\left|\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right| \\
&\qquad -\operatorname{Re}\left\{\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right\} \\
&\qquad \le (1+\beta)\left|\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right| \\
&\qquad \le \frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)\left[1+\gamma(k-1)\right]|a_k|\,b_k}{1-\sum\limits_{k=2}^{\infty}\left[1+\gamma(k-1)\right]|a_k|\,b_k}.
\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]|a_k|\,b_k \le 1-\alpha,$$

and hence the proof is completed. □

Theorem 2. *A necessary and sufficient condition for $f(z)$ of the form (1.8) to be in the class $TS_\gamma(f,g;\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]a_kb_k \le 1-\alpha, \tag{2.2}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f(z)\in TS_\gamma(f,g;\alpha,\beta)$ and z is real, then

$$\frac{1-\sum\limits_{k=2}^{\infty}k\left[1+\gamma(k-1)\right]a_kb_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left[1+\gamma(k-1)\right]a_kb_kz^{k-1}}-\alpha \ge \beta\left|\frac{\sum\limits_{k=2}^{\infty}(k-1)\left[1+\gamma(k-1)\right]a_kb_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left[1+\gamma(k-1)\right]a_kb_kz^{k-1}}\right|.$$

Letting $z\to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]a_kb_k \le 1-\alpha.$$

□

Corollary 1. *Let the function $f(z)$ be defined by (1.8) be in the class $TS_\gamma(f,g;\alpha,\beta)$. Then*

$$a_k \le \frac{1-\alpha}{\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]b_k} \quad (k\ge 2). \tag{2.3}$$

The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} z^k \ (k \geq 2). \tag{2.4}$$

3. Distortion theorems

Theorem 3. *Let the function $f(z)$ be defined by (1.8) be in the class $TS_\gamma(f,g;\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \geq r - \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2} r^2 \tag{3.1}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2} r^2, \tag{3.2}$$

provided that $b_k \geq b_2$ $(k \geq 2)$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2} z^2, \tag{3.3}$$

at $z = r$ and $z = re^{i(2k+1)\pi}$ $(k \in Z)$.

Proof. Since for $k \geq 2$,

$$(2-\alpha+\beta)(1+\gamma)b_2 \leq [k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k,$$

using Theorem 2, we have

$$\begin{aligned}(2-\alpha+\beta)(1+\gamma)b_2 &\sum_{k=2}^{\infty} a_k \\ &\leq \sum_{k=2}^{\infty} [k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,a_k b_k \leq 1-\alpha\end{aligned} \tag{3.4}$$

that is, that

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2}. \tag{3.5}$$

From (1.8) and (3.5), we have

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2} r^2 \tag{3.6}$$

and

$$|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2} r^2. \tag{3.7}$$

This completes the proof of Theorem 3. □

Theorem 4. *Let the function $f(z)$ be defined by (1.8) be in the class $TS_\gamma(f,g;\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$\left|f'(z)\right| \geq 1-\frac{2(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2}r \tag{3.8}$$

and

$$\left|f'(z)\right| \leq 1+\frac{2(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2}r, \tag{3.9}$$

provided that $b_k \geq b_2$ $(k \geq 2)$. The result is sharp for the function $f(z)$ given by (3.3).

Proof. From Theorem 2 and (3.5), we have

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2}. \tag{3.10}$$

Since the remaining part of the proof is similar to the proof of Theorem 3, we omit the details. □

4. Convex linear combinations

Theorem 5. *Let $\mu_\upsilon \geq 0$ for $\upsilon = 1,2,\ldots,l$ and $\sum_{\upsilon=1}^{l}\mu_\upsilon \leq 1$. If the functions $F_\upsilon(z)$ defined by*

$$F_\upsilon(z) = z-\sum_{k=2}^{\infty} a_{k,\upsilon} z^k \quad (a_{k,\upsilon}\geq 0;\ \upsilon=1,2,\ldots,l) \tag{4.1}$$

are in the class $TS_\gamma(f,g;\alpha,\beta)$ for every $\upsilon=1,2,\ldots,l$, then the function $f(z)$ defined by

$$f(z) = z-\sum_{k=2}^{\infty}\left(\sum_{\upsilon=1}^{l}\mu_\upsilon a_{k,\upsilon}\right)z^k$$

is in the class $TS_\gamma(f,g;\alpha,\beta)$

Proof. Since $F_\upsilon(z)\in TS_\gamma(f,g;\alpha,\beta)$, it follows from Theorem 2 that

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]a_{k,\upsilon}b_k \leq 1-\alpha, \tag{4.2}$$

for every $\upsilon = 1, 2, \ldots, l$. Hence

$$\sum_{k=2}^{\infty} [k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)] \left(\sum_{\upsilon=1}^{l} \mu_\upsilon a_{k,\upsilon}\right) b_k$$
$$= \sum_{\upsilon=1}^{l} \mu_\upsilon \left(\sum_{k=2}^{\infty} [k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\, a_{k,\upsilon} b_k\right)$$
$$\leq (1-\alpha)\sum_{\upsilon=1}^{l} \mu_\upsilon \leq 1-\alpha.$$

By Theorem 2, it follows that $f(z) \in TS_\gamma(f, g; \alpha, \beta)$. □

Corollary 2. *The class $TS_\gamma(f, g; \alpha, \beta)$ is closed under convex linear combinations.*

Theorem 6. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\, b_k} z^k \quad (k \geq 2) \tag{4.3}$$

for $-1 \leq \alpha < 1$, $0 \leq \gamma \leq 1$ and $\beta \geq 0$. Then $f(z)$ is in the class $TS_\gamma(f, g; \alpha, \beta)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{4.4}$$

where $\mu_k \geq 0$ and $\sum\limits_{k=1}^{\infty} \mu_k = 1$.

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \tag{4.5}$$
$$= z - \sum_{k=2}^{\infty} \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\, b_k} \mu_k z^k.$$

Then it follows that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\, b_k}{1-\alpha} \tag{4.6}$$
$$\times \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\, b_k} \mu_k = \sum_{k=2}^{\infty} \mu_k = 1-\mu_1 \leq 1.$$

So, by Theorem 2, $f(z) \in TS_\gamma(f, g; \alpha, \beta)$.

Conversely, assume that the function $f(z)$ defined by (1.8) belongs to the class $TS_\gamma(f, g; \alpha, \beta)$. Then

$$a_k \leq \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} \quad (k \geq 2)\,. \tag{4.7}$$

Setting

$$\mu_k = \frac{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,a_k b_k}{1-\alpha} \quad (k \geq 2) \tag{4.8}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{4.9}$$

we can see that $f(z)$ can be expressed in the form (4.4). This completes the proof of Theorem 6. $\square$

Corollary 3. *The extreme points of the class $TS_\gamma(f, g; \alpha, \beta)$ are the functions $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} z^k \quad (k \geq 2)\,.$$

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 7. *Let the function $f(z)$ defined by (1.8) be in the class $TS_\gamma(f, g; \alpha, \beta)$. Then $f(z)$ is close-to-convex of order $\rho\,(0 \leq \rho < 1)$ in $|z| < r_1$, where*

$$r_1 = \inf_{k \geq 2} \left\{ \frac{(1-\rho)\,[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}{k(1-\alpha)} \right\}^{\frac{1}{k-1}}. \tag{5.1}$$

The result is sharp, the extremal function being given by (2.4).

Proof. We must show that

$$\left|f'(z) - 1\right| \leq 1-\rho \text{ for } |z| < r_1,$$

where r_1 is given by (5.1). Indeed we find from the definition (1.8) that

$$\left|f'(z) - 1\right| \leq \sum_{k=2}^{\infty} k a_k \left|z\right|^{k-1}.$$

Thus

$$\left|f'(z) - 1\right| \leq 1-\rho,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k \left|z\right|^{k-1} \leq 1. \tag{5.2}$$

But, by Theorem 2, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}{1-\alpha},$$

that is, if

$$(5.3) \quad |z| \leq \left\{\frac{(1-\rho)\,[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}{k(1-\alpha)}\right\}^{\dfrac{1}{k-1}} \quad (k \geq 2).$$

Theorem 7 follows easily from (5.3). □

Theorem 8. *Let the function $f(z)$ defined by (1.8) be in the class $TS_\gamma(f,g;\alpha,\beta)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$(5.4) \qquad r_2 = \inf_{k\geq 2}\left\{\frac{(1-\rho)\,[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}{(k-\rho)\,(1-\alpha)}\right\}^{\dfrac{1}{k-1}}.$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho \text{ for } |z| < r_2,$$

where r_2 is given by (5.4). Indeed we find, again from the definition (1.8) that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k\,|z|^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho$$

if

$$(5.5) \qquad \sum_{k=2}^{\infty}\frac{(k-\rho)a_k\,|z|^{k-1}}{(1-\rho)} \leq 1.$$

But, by Theorem 2, (5.5) will be true if

$$\frac{(k-\rho)\,|z|^{k-1}}{(1-\rho)} \leq \frac{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}{(1-\alpha)}$$

that is, if

$$(5.6) \quad |z| \leq \left\{\frac{(1-\rho)\,[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}{(k-\rho)\,(1-\alpha)}\right\}^{\dfrac{1}{k-1}} (k \geq 2).$$

Theorem 8 follows easily from (5.6). □

Corollary 4. *Let the function $f(z)$ defined by (1.8) be in the class $TS_{\gamma}(f, g; \alpha, \beta)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_{k \geq 2} \left\{ \frac{(1-\rho)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right] b_k}{k\,(k-\rho)\,(1-\alpha)} \right\}^{\frac{1}{k-1}}. \tag{5.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

6. A family of integral operators

In view of Theorem 2, we see that $z - \sum_{k=2}^{\infty} d_k z^k$ is in $TS_{\gamma}(f, g; \alpha, \beta)$ as long as $0 \leq d_k \leq a_k$ for all k. In particular, we have

Theorem 9. *Let the function $f(z)$ defined by (1.8) be in the class $TS_{\gamma}(f, g; \alpha, \beta)$ and c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \tag{6.1}$$

also belongs to the class $TS_{\gamma}(f, g; \alpha, \beta)$.

Proof. From the represtation (6.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} d_k z^k,$$

where

$$d_k = \left(\frac{c+1}{c+k} \right) a_k \leq a_k \;\; (k \geq 2)\,.$$

On the other hand, the converse is not true. This leads to a radius of univalence result. □

Theorem 10. *Let the function $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$ $(a_k \geq 0)$ be in the class $TS_{\gamma}(f, g; \alpha, \beta)$, and let c be a real number such that $c > -1$. Then the function $f(z)$ given by (6.1) is univalent in $|z| < R^{\star}$, where*

$$R^{\star} = \inf_{k \geq 2} \left\{ \frac{(c+1)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right] b_k}{k\,(c+k)\,(1-\alpha)} \right\}^{\frac{1}{k-1}}. \tag{6.2}$$

The result is sharp.

Proof. From (6.1), we have

$$f(z) = \frac{z^{1-c}\left[z^c F(z)\right]'}{(c+1)} = z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1} \right) a_k z^k \, (c > -1)\,.$$

In order to obtain the required result, it suffices to show that

$$\left|f^{'}(z)-1\right|<1 \text{ wherever } \quad |z|<R^{\star},$$

where $R^{\star}$ is given by (6.2). Now

$$\left|f^{'}(z)-1\right| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k \left|z\right|^{k-1}.$$

Thus $\left|f^{'}(z)-1\right|<1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k \left|z\right|^{k-1}<1. \tag{6.3}$$

But Theorem 2 confirms that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]\left[1+\gamma(k-1)\right] a_k b_k}{1-\alpha} \leq 1. \tag{6.4}$$

Hence (6.3) will be satisfied if

$$\frac{k(c+k)}{(c+1)}\left|z\right|^{k-1}<\frac{[k(1+\beta)-(\alpha+\beta)]\left[1+\gamma(k-1)\right] b_k}{(1-\alpha)},$$

that is, if

$$|z|<\left[\frac{(c+1)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right] b_k}{k(c+k)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.5}$$

Therefore, the function $f(z)$ given by (6.1) is univalent in $|z|<R^{\star}$. Sharpness of the result follows if we take

$$f(z)=z-\frac{(c+k)(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]\left[1+\gamma(k-1)\right] b_k\,(c+1)} z^k \ (k \geq 2). \tag{6.6}$$

□

7. Partial sums

Following the earlier works by Silverman [12] and Siliva [13] on partial sums of analytic functions, we consider in this section partial sums of functions in the class $TS_{\gamma}(f,g;\alpha,\beta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_n\ (z)$ and $f^{'}(z)$ to $f_n^{'}\ (z)$.

Theorem 11. *Define the partial sums $f_1(z)$ and $f_n(z)$ by*

$$f_1(z)=z \text{ and } f_n(z)=z+\sum_{k=2}^{n} a_k z^k, \quad (n \in N\backslash\{1\}).$$

Let $f(z) \in TS_\gamma(f, g; \alpha, \beta)$ be given by (1.1) and satisfies the condition (2.2) and

$$c_k \geq \begin{cases} 1, & k = 2, 3, \ldots, n, \\ c_{n+1}, & k = n+1, n+2, \ldots, \end{cases} \tag{7.1}$$

where, for convenience,

$$c_k = \frac{[k(1+\beta) - (\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}{1-\alpha}. \tag{7.2}$$

Then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{c_{n+1}} \ (z \in U;\ n \in N) \tag{7.3}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{c_{n+1}}{1+c_{n+1}}. \tag{7.4}$$

Proof. For the coefficients c_k given by (7.2) it is not difficult to verify that

$$c_{k+1} > c_k > 1. \tag{7.5}$$

Therefore we have

$$\sum_{k=2}^{n} |a_k| + c_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} c_k |a_k| \leq 1. \tag{7.6}$$

By setting

$$g_1(z) = c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{n+1}}\right)\right\} = 1 + \frac{c_{n+1}\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=2}^{n} a_k z^{k-1}} \tag{7.7}$$

and applying (7.6), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{c_{n+1}\sum\limits_{k=n+1}^{\infty} |a_k|}{2 - 2\sum\limits_{k=2}^{n} |a_k| - c_{n+1}\sum\limits_{k=n+1}^{\infty} |a_k|}. \tag{7.8}$$

Now

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1$$

if

$$\sum_{k=2}^{n} |a_k| + c_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq 1.$$

From the condition (2.2), it is sufficient to show that

$$\sum_{k=2}^{n}|a_k| + c_{n+1}\sum_{k=n+1}^{\infty}|a_k| \le \sum_{k=2}^{\infty}c_k\,|a_k|$$

which is equivalent to

$$\sum_{k=2}^{n}(c_k-1)\,|a_k| + \sum_{k=n+1}^{\infty}(c_k-c_{n+1})\,|a_k| \ge 0, \tag{7.9}$$

which readily yields the assertion (7.3) of Theorem 11. In order to see that

$$f(z) = z + \frac{z^{n+1}}{c_{n+1}} \tag{7.10}$$

gives sharp result, we observe that for $z = re^{\frac{i\pi}{n}}$ that $\dfrac{f(z)}{f_n(z)} = 1 + \dfrac{z^n}{c_{n+1}} \to 1 - \dfrac{1}{c_{n+1}}$ as $z \to 1^-$. Similarly, if we take

$$g_2(z) = (1+c_{n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1+c_{n+1}}\right\} = 1 - \frac{(1+c_{n+1})\sum\limits_{k=n+1}^{\infty}a_k z^{k-1}}{1+\sum\limits_{k=2}^{\infty}a_k z^{k-1}} \tag{7.11}$$

and making use of (7.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1+c_{n+1})\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{n}|a_k| - (1-c_{n+1})\sum\limits_{k=n+1}^{\infty}|a_k|} \tag{7.12}$$

which leads us immediately to the assertion (7.4) of Theorem 11.

The bound in (7.4) is sharp for each $n \in N$ with the extremal function $f(z)$ given by (7.10). The proof of Theorem 11 is thus complete. □

Theorem 12. *If $f(z)$ of the form (1.1) satisfies the condition (2.2). Then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\} \ge 1 - \frac{n+1}{c_{n+1}}, \tag{7.13}$$

and

$$\operatorname{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\} \ge \frac{c_{n+1}}{n+1+c_{n+1}}, \tag{7.14}$$

where c_k defined by (7.2) and satisfies the condition

$$c_k \ge \begin{cases} k, & \text{if } k = 2,3,\dots,n, \\ \dfrac{c_{n+1}}{n+1}k, & \text{if } k = n+1, n+2, \dots. \end{cases} \tag{7.15}$$

The results are sharp with the function $f(z)$ given by (7.10).

Proof. By setting

$$
\begin{aligned}
g(z) &= \frac{c_{n+1}}{n+1}\left\{\frac{f'(z)}{f_n'(z)} - \left(1 - \frac{n+1}{c_{n+1}}\right)\right\} \\
&= \frac{1 + \frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k a_k z^{k-1} + \sum_{k=2}^{n} k a_k z^{k-1}}{1 + \sum_{k=2}^{n} k a_k z^{k-1}} \qquad (7.16) \\
&= 1 + \frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^{n} k a_k z^{k-1}}.
\end{aligned}
$$

Then

$$
\left|\frac{g(z)-1}{g(z)+1}\right| \le \frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k\,|a_k|}{2 - 2\sum_{k=2}^{n} k\,|a_k| - \frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k\,|a_k|}. \tag{7.17}
$$

Now

$$
\left|\frac{g(z)-1}{g(z)+1}\right| \le 1,
$$

if

$$
\sum_{k=2}^{n} k\,|a_k| + \frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k\,|a_k| \le 1, \tag{7.18}
$$

since the left hand side of (7.18) is bounded above by $\sum_{k=2}^{\infty} c_k\,|a_k|$ if

$$
\sum_{k=2}^{n}(c_k - k)\,|a_k| + \sum_{k=n+1}^{\infty}(c_k - \frac{c_{n+1}}{n+1}k)\,|a_k| \ge 0 \tag{7.19}
$$

and the proof of (7.13) is complete.

To prove the result (7.14), define the function $g(z)$ by

$$
\begin{aligned}
g(z) &= \left(\frac{n+1+c_{n+1}}{n+1}\right)\left\{\frac{f_n'(z)}{f'(z)} - \frac{c_{n+1}}{n+1+c_{n+1}}\right\} \\
&= 1 - \frac{\left(1 + \frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} k a_k z^{k-1}},
\end{aligned}
$$

and making use of (7.19), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{c_{n+1}}{n+1}\right)\sum\limits_{k=n+1}^{\infty} k\,|a_k|}{2-2\sum\limits_{k=2}^{n} k\,|a_k| - \left(1+\frac{c_{n+1}}{n+1}\right)\sum\limits_{k=n+1}^{\infty} k\,|a_k|} \leq 1,$$

which leads us immediately to the assertion (7.14) of Theorem 12. □

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