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ALMOST GEODESIC MAPPINGS ONTO GENERALIZED RICCI-SYMMETRIC MANIFOLDS

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Abstract. Our aim is to continue investigations concerning existence of almost geodesic mappings of manifolds with linear connection. We deduce necessary and sufficient conditions for existence of the so-called canonical almost geodesic mappings of type π of a manifold endowed with a linear connection onto generalized Ricci-symmetric manifolds. Our result is a generalization of some previous results by N. S. Sinyukov.

1. Introduction

First let us recall the main concepts and terminology. Let (M, ∇) be a smooth (C^∞) n-dimensional manifold endowed with a linear connection ∇. Let TM denote the tangent bundle of M, let $p_M : TM \to M$ be the natural projection, and let $\Lambda^2 TM$ denote the associated vector bundle of bivectors. $\mathcal{X}(M)$ denotes the $\mathcal{F}(M)$-module of vector fields on M over the ring $\mathcal{F}(M)$ of smooth functions on M. If $f\colon M \to \bar{M}$ is a diffeomorphism then $Tf\colon TM \to T\bar{M}$ is the corresponding tangent mapping, or differential, $Tf = f_*$. Unless otherwise specified, all objects under consideration are supposed to be differentiable of a sufficiently high class.

1.1. **Vector fields and distributions parallel along a curve.** Recall that an n-dimensional distribution on an open neighborhood $U \subseteq M$ $(\dim M = m \geq n)$ is a map $D\colon U \to TU$, $U \ni x \mapsto D_x \subseteq T_xM$, and D is called differentiable of the class C^k if it admits a local C^k-basis around any point. In short, $D = \operatorname{span}(X_1, \dots, X_n)$ on V. A C^k-vector field X (on U) belongs to D, $X \in D$, if $X_x \in D_x$ for all $x \in U$.

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Let $c\colon I \to M$, $t \mapsto c(t)$, with $I \subset \mathbb{R}$ being an open interval, denote a (C^k-, or smooth) curve on M. Let ξ denote the corresponding (C^{k-1}, or smooth) tangent ("velocity") vector field along c, $\xi(t) = \left(c(t), \frac{dc(t)}{dt}\right)$, $t \in I$. In the following, we will consider only those curves which are *regular* in the sense that the tangent vector field ξ along c does not vanish on the definition domain I, that is, $c'(t) = \frac{dc(t)}{dt} \neq 0$ for all $t \in I$. Besides the velocity field ξ, let us introduce vector fields ξ_1, ξ_2, associated to a curve c, by the formula

$$\xi_1 = \nabla_\xi \xi, \qquad \xi_2 = \nabla_\xi \xi_1 \tag{1}$$

Under a (C^k-)*vector field along* c we mean a (C^k-)mapping $Y\colon I \to TM$ such that $p_M \circ Y = c$, that is, $Y(t) \in T_{c(t)}M$ for any $t \in I$. Similarly, a differentiable n-distribution along c can be introduced as a span of an n-tuple of (differentiable) vector fields along c. The velocity field $\xi(t)$ generates a one-dimensional distribution Ξ along c. Remark that any differentiable vector field (differentiable distribution, respectively) along c can be extended into a differentiable vector field (distribution) on some neighborhood U of $c(I)$.

Denote by $\tau_{c(t_0),c(t)}$ the parallel transport along c relative to ∇ from $c(t_0)$ to $c(t)$. A vector field Y along c on (M,∇) is called *parallel* along c relative to the given connection if $\nabla_\xi Y = 0$. A distribution D (defined along c, or on some open neighborhood of $c(I)$) is called *parallel along* c if for any $t_0 \in I$ and any vector $X_0 \in D_{c(t_0)}$, the image of X_0 under the parallel translation τ along c (from the point $c(t_0)$ to $c(t)$) belongs to D, $\tau_{c(t_0),c(t)}X_0 \in D_{c(t)}$ for all $t \in I$. A distribution D parallel along c admits a (local) basis parallel along c; parallelism along c is independent on reparametrizations of the path.

Lemma 1. *Let D be a two-dimensional distribution along c. Let X_1, X_2 be vector fields along c which form a basis of D; $D = \operatorname{span}(X_1, X_2)$. Then the necessary and sufficient condition for D to be parallel along c may be expressed as follows: there are real function $a_i^j\colon I \to \mathbb{R}$ of the parameter t such that*

$$\nabla_\xi X_i = a_i^j X_j, \qquad i,j \in \{1,2\} \tag{2}$$

hold (covariant derivatives along c of basis vector fields belong to the distribution, [5, p. 4].

1.2. **Almost geodesic curves.** Let (M,∇) be a smooth manifold endowed with a linear connection. Let $c\colon I \to M$ be a smooth regular curve on M. Recall that c is called a *geodesic curve* (in short, g.c.), under a general parametrization, if for any initial value $t_0 \in I$ of the parameter, the vector field $\tau_{c(t_0),c(t)}(\xi(t_0))$ along c arising from images of $\xi(t_0)$ under the parallel propagation along c, belongs to the 1-dimensional distribution $\Xi = \operatorname{span}(\xi)$ along c (generated by the velocity field). Hence the vectors $\xi_1(t)$ and $\xi(t)$ are collinear for any $t \in I$ if and only if c is a geodesic curve. Equivalently, c is a g. c. if and only if ξ is recurrent along

c which means: there is a real function $\lambda(t)\colon I \to \mathbb{R}$ such that the formula

$$\nabla_{\xi(t)}\xi(t) = \lambda(t)\xi(t) \tag{3}$$

holds. If the curve is parametrized by canonical (affine) parameter, the condition (3) for geodesic curves take the usual form $\nabla_{\xi(s)}\xi(s) = 0$ for $\in I$, and we speak about geodesics.

Geodesic curves can be naturally generalized as follows. According to [5], we call c *almost geodesic* if there is a 2-dimensional (differentiable) distribution D (along c, or on some neighborhood of $c(I)$) parallel along c relative to ∇ such that the tangent vector field ξ belongs to D (or Ξ is a subdistribution of D), $\xi(t) \in D_{c(t)}$ for $t \in I$. Equivalently, c is almost geodesic if and only if there exist vector fields X_1, X_2 along c satisfying (2), i.e. parallel, and (differentiable) real functions $b^i(t)$, $t \in \mathbb{R}$, defined along c, such that $\xi = b^1X_1 + b^2X_2$ holds. Obviously, geodesic curves, particularly geodesics, can serve as examples of almost geodesic curves.

For almost geodesic curves, ξ_1 and ξ_2 from (1) belong to the corresponding distribution D. If the vector fields ξ and ξ_1 are independent at any point (the (local) curve c is not a geodesic one), then $D = \operatorname{span}(\xi, \xi_1)$. So we can easily check that another equivalent characterization is:

Lemma 2. *A curve is almost geodesic if and only if* $\xi_2 \in \operatorname{span}(\xi, \xi_1)$.

2. Almost geodesic mappings

The concept of an almost geodesic mapping was introduced by V. M. Chernyshenko [3], and later on by N. S. Sinyukov, from a rather different point of view, [5, 6, 7, 8]. The theory of almost geodesic mappings in a developed form can be found in [5, 6, 7, 8].

Let (M, ∇), $(\bar{M}, \bar{\nabla})$ be smooth n-dimensional manifolds, $n > 2$, endowed with torsion-free linear connection.

Definition. [5, 6, 7, 8] A diffeomorphism $f\colon M \to \bar{M}$ is called *almost geodesic* if any geodesic curve of (M, ∇) is mapped under f onto an almost geodesic curve in $(\bar{M}, \bar{\nabla})$.

Conventions. From now on, all connections under consideration are torsion-free ($\equiv$ symmetric). If $f\colon M \to \bar{M}$ is a diffeomorphism we always suppose that the connections ∇ and $\bar{\nabla}$ are defined on the same manifold M, and we may in fact assume diffeomorphisms $f\colon (M, \nabla) \to (M, \bar{\nabla})$, which is more convenient from the technical reasons: we can make use of the well-known fact that two linear connections ∇ and $\bar{\nabla}$ on the same manifold M always differ up to a $(1, 2)$-tensor field P,

$$\bar{\nabla}(X, Y) = \nabla(X, Y) + P(X, Y), \qquad X, Y \in \mathcal{X}(M), \tag{4}$$

and if the connections are symmetric, then P is also symmetric in X, Y. Moreover, we always identify a given curve c with its image $\bar{c} = f \circ c$, similarly we identify the tangent vector function $\xi(t)$ with the corresponding vector function $\bar{\xi}(t) = Tf(\xi(t))$. Given a diffeomorphism $f\colon (M,\nabla) \to (M,\bar{\nabla})$ then P determined by (4) will be called here the *deformation tensor* of the given connections under f ([6]). For a deformation tensor P (of type $(1,2)$), let us introduce a new tensor field (of type $(1,3)$, denoted by the same symbol) by

$$P(X,Y,Z) = \sum_{CS(X,Y,Z)} \nabla_Z P(X,Y) + P(P(X,Y),Z), \quad X,Y,Z \in \mathcal{X}(M)$$

where $\sum_{CS(\,,\,,\,)}$ means the cyclic sum on arguments in brackets (i. e. symmetrization without coefficients). Let $X \wedge Y$ means a decomposable bivector, an exterior product of X and Y. A diffeomorphism $f\colon (M,\nabla) \to (M,\bar{\nabla})$ is almost-geodesic if and only if the deformation tensor P satisfies

$$\text{(5)} \qquad P(X_1,X_2,X_3) \wedge P(X_4,X_5) \wedge X_6 = 0 \text{ for all } X_i \in \mathcal{X}(M),\ i = 1,\dots,6.$$

In local coordinates, (5) reads $P^{[h}_{(pqr}P^{i}_{su}\delta^{j]}_{v)} = 0$ where the round and square brackets denote symmetrization and alternation of indices, respectively.

3. Classification of almost geodesic mappings

N.S. Sinyukov distinguished three kinds of almost geodesic mappings, [5, 6], namely π_1, π_2, and π_3, characterized, respectively, by the conditions for the deformation tensor:

$$\pi_1\colon P(X,X,X) + P(P(X,X),X) = a(X,X){\cdot}X + b(X){\cdot}P(X,X),\ X \in \mathcal{X}(M),$$

where a is a symmetric type $(0,2)$ tensor field and b is a one-form;

$$\pi_2\colon P(X,X) = \psi(X)\cdot X + \varphi(X)\cdot F(X), \quad X \in \mathcal{X}(M),$$

where ψ and ϕ are one-forms, and F is a type $(1,1)$ tensor field satisfying

$$(\nabla F)(X;X) + F(F(X),X) = \mu(X)\cdot X + \varrho(X)\cdot F(X), \quad X \in \mathcal{X}(M)$$

for some one-forms μ, ϱ;

$$\pi_3\colon P(X,X) = \psi(X)\cdot X + a(X,X)\cdot Z, \quad X \in \mathcal{X}(M)$$

where ψ is a one-form, a is a symmetric bilinear form and $Z \in \mathcal{X}(M)$ is a vector field satisfying

$$(\nabla Z)(X) = h\cdot X + \theta(X)\cdot Z$$

for some scalar function $h\colon M \to \mathbb{R}$ and some one-form θ.

The so-called $\tilde{\pi}_1$-*mappings*, *canonical* almost geodesic mappings, are characterized among almost geodesic mappings by the condition $b = 0$ on the right hand side. That is, the deformation tensor of a $\tilde{\pi}_1$-mapping satisfies

$$\text{(6)} \qquad P(X,X,X) + P(P(X,X),X) = a(X,X)\cdot X, \quad X \in \mathcal{X}(M).$$

It is known that any π_1-mapping arises as a composition of a $\tilde{\pi}_1$-mapping and a geodesic one. But geodesic mappings can be considered as trivial almost geodesic mappings, and can be omitted in our further considerations; they have been analysed and classified in [1]. Our aim is to study $\tilde{\pi}_1$-mappings of affine manifolds onto particular types of Riemannian spaces, namely those cases that induce integrable systems.

4. Ricci-symmetric and generalized Ricci-symmetric manifolds

Under a *Ricci-symmetric manifold (space)* we mean a manifold (M, ∇) with linear connection (a pseudo-Riemannian space (M, g), respectively) for which the Ricci tensor is parallel (=covariantly constant),

$$\nabla \mathrm{Ric} = 0.$$

It was proven in [6] that the family of all $\tilde{\pi}_1$-mappings of a manifold (M, ∇) (= "affine manifold") onto Ricci-symmetric (pseudo-)Riemannian spaces $(\bar{M}, \bar{g})$ ($\bar{\nabla}\bar{\mathrm{Ric}} = 0$) is given by an integrable system of differentiable equations (in covariant derivatives). Consequently, given a manifold with a symmetric connection, the family of all Ricci-symmetric Riemannian spaces $(\bar{M}, \bar{g})$ which can serve as images of the given manifold (M, ∇) under some $\tilde{\pi}_1$-mapping, depends on a finite set of parameters.

On the other hand, geodesic mappings form a subset in the set of $\tilde{\pi}_1$-mappings; they obey the definition. But basic equations describing geodesic mappings of a manifold with linear connection do not form an integrable system of Cauchy type, since a general solution depends on n arbitrary functions. It follows that the conditions (6) describing $\tilde{\pi}_1$-mappings (i.e. canonical almost geodesic mappings) of affine manifolds do not, in general, induce an integrable system.

In the following, we consider a particular case when (6) can be transformed into an integrable system, generalizing the results of Sinyukov. Namely, we will investigate $\tilde{\pi}_1$-mappings of an affine manifold (M, ∇) onto the so-called generalized Ricci-symmetric manifolds.

An affine manifold (M, ∇) will be called a *generalized Ricci-symmetric manifold* if its Ricci tensor satisfies

$$\nabla \mathrm{Ric}\,(Y, Z; X) + \nabla \mathrm{Ric}\,(X, Z; Y) = 0, \tag{7}$$

that is, $\nabla_X \mathrm{Ric}\,(Y, Z) = -\nabla_Y \mathrm{Ric}\,(X, Z)$. We do not a priori suppose the Ricci tensor be symmetric. If Ric is symmetric and (7) holds then Ric is parallel, $\nabla \mathrm{Ric} = 0$, and (M, ∇) is a Ricci-symmetric manifold. Einstein spaces (Riemannian spaces characterized by the property that the Ricci tensor is proportional to the metric tensor) satisfy (7) since they satisfy $\nabla \mathrm{Ric} = 0$, hence are generalized Ricci-symmetric. In this sense, generalized Ricci-symmetric spaces can be considered as a certain generalization of Einstein spaces.

5. Almost geodesic mappings $\tilde{\pi}_1$ onto generalized Ricci-symmetric manifolds

Given affine m-dimensional manifolds $\mathbb{A} = (M, \nabla)$ and $\bar{\mathbb{A}} = (\bar{M}, \bar{\nabla})$ with the corresponding curvature tensors R and $\bar{R}$, respectively, all connection-preserving mappings $f\colon M \to \bar{M}$ can be described by the following system of (differential) equations, [6, 7, 8]:

$$3(\nabla_Z P(X,Y) + P(Z, P(X,Y))) \\ = \sum_{CS(X,Y)} (R(Y,Z)X - \bar{R}(Y,Z)X) + \sum_{CS(X,Y,Z)} a(X,Y)Z.$$

It becomes clear that the above invariant formulas are rather complicated. As for the rest, we prefer to express our equalities in local coordinates (with respect to a map (U, φ) on M). This formulas have the following local expression

$$3(P^h_{ij,k} + P^h_{k\alpha} P^\alpha_{ij}) = R^h_{(ij)k} - \bar{R}^h_{(ij)k} + a_{(ij}\delta^h_{k)}, \tag{8}$$

where P^h_{ij}, a_{ij}, R^h_{ijk}, $\bar{R}^h_{ijk}$ are local components of tensors P, R, $\bar{R}$ and a, respectively, δ^h_k is the Kronecker delta, "," denotes covariant derivative with respect to ∇.

The system (8) can be considered as a system of partial differential equations for functions P^h_{ij} on M, i.e. for components of the deformation tensor; the corresponding integrability conditions are

$$\bar{R}^h_{(ij)[k,\ell]} = R^h_{(ij)[k,\ell]} + \delta^h_{(i} a_{jk),\ell} - \delta^h_{(i} a_{j\ell),k} - 3(-P^\alpha_{ij}\bar{R}^h_{\alpha k\ell} + P^h_{\alpha(j} R^\alpha_{i)k\ell}) \\ - P^h_{\alpha k}(R^\alpha_{(ij)\ell} - \bar{R}^\alpha_{(ij)\ell}\delta^\alpha_{(i} a_{j\ell)}) + P^h_{\alpha\ell}(R^\alpha_{(ij)k} - \bar{R}^\alpha_{(ij)k}\delta^\alpha_{(i} a_{jk)}).$$

Passing from $\nabla\bar{R}$ to $\bar{\nabla}\bar{R}$ on the left hand side we get the following integrability conditions of the system (8):

$$\bar{R}^h_{(ij)[k;\ell]} = \delta^h_{(i} a_{jk),\ell} - \delta^h_{(i} a_{j\ell),k} + \Theta^h_{ijk\ell}\,, \tag{9}$$

where

$$\Theta^h_{ijk\ell} = R^h_{(ij)[k,\ell]} - 3(-P^\alpha_{ij}\bar{R}^h_{\alpha k\ell} + P^h_{\alpha(j} R^\alpha_{i)k\ell}) \\ - P^h_{\alpha k}(R^\alpha_{(ij)\ell} - \bar{R}^\alpha_{(ij)\ell}\delta^\alpha_{(i} a_{j\ell)}) + P^h_{\alpha\ell}(R^\alpha_{(ij)k} - \bar{R}^\alpha_{(ij)k}\delta^\alpha_{(i} a_{jk)}) \\ - P^\alpha_{\ell(i}\bar{R}^h_{|\alpha|j)k} - P^\alpha_{\ell(i}\bar{R}^h_{j)\alpha k} + P^\alpha_{k(i}\bar{R}^h_{|\alpha|j)\ell} + P^\alpha_{k(i}\bar{R}^h_{j)\alpha\ell}.$$

Using the Bianchi identity we can write (9) in local coordinate as

$$\bar{R}^h_{i\ell k;j} + \bar{R}^h_{j\ell k;i} = \delta^h_{(i} a_{jk),\ell} - \delta^h_{(i} a_{j\ell),k} + \Theta^h_{ijk\ell}\,,$$

where ";" denotes covariant derivative with respect to $\bar{\nabla}$. Contraction in h and k gives the following equality for covariant derivatives of components of the Ricci tensor $\bar{\mathrm{Ric}}$ of $\bar{\nabla}$:

$$\bar{R}_{i\ell;j} + \bar{R}_{j\ell;i} = (n+1)a_{ij,\ell} - a_{\ell(i,j)} + \Theta^\alpha_{ij\alpha\ell}\,. \tag{10}$$

In the following let us suppose that the affine manifold $(\bar{M}, \bar{\nabla})$ is a generalized Ricci-symmetric space, that is, (7) holds. In local coordinates, (7) reads

$$\bar{R}_{ij;k} + \bar{R}_{kj;i} = 0.$$

Under this assumption, (10) reads

$$(n+1)a_{ij,\ell} - a_{\ell i,j} - a_{\ell j,i} = -\Theta^{\alpha}_{ij\alpha\ell}. \tag{11}$$

Using symmetrization in ℓ, i gives

$$a_{\ell i,j} + a_{\ell j,i} = -\frac{1}{n}\Theta^{\alpha}_{(i|\ell\alpha|j)} + \frac{2}{n}a_{ij,\ell}.$$

Now (11) reads

$$\frac{n^2+n-2}{n}\, a_{ij,\ell} = -\,\Theta^{\alpha}_{ij\alpha\ell} - \frac{1}{n}\Theta^{\alpha}_{(i|\ell\alpha|j)}. \tag{12}$$

Applying covariant differentiation with respect to $\bar{\nabla}$ to the integrability conditions (9), followed by passing from covariant derivative $\bar{\nabla}$ to ∇ on the right hand side, we get

$$\bar{R}^h_{(ij)k;\ell m} - \bar{R}^h_{(ij)\ell;mk} = \delta^h_{(i}a_{jk),\ell m} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk\ell m}\,, \tag{13}$$

where

$$\begin{aligned}T^h_{ijk\ell m} = {} & \bar{R}^h_{\alpha mk}\bar{R}^{\alpha}_{(ij)\ell} - \bar{R}^{\alpha}_{\ell mk}\bar{R}^h_{(ij)\alpha} - \bar{R}^{\alpha}_{jmk}\bar{R}^h_{(i\alpha)\ell} - \bar{R}^{\alpha}_{imk}\bar{R}^h_{(j\alpha)\ell}\\
& - P^h_{m\alpha}\delta^{\alpha}_{(i}a_{jk),\ell} - P^{\alpha}_{mj}\delta^h_{(i}a_{\alpha k),\ell} - P^{\alpha}_{mi}\delta^h_{(\alpha}a_{jk),\ell} - P^{\alpha}_{mk}\delta^h_{(\alpha}a_{ij),\ell} - P^{\alpha}_{ml}\delta^h_{(i}a_{jk),\alpha}\\
& - P^h_{m\alpha}\delta^{\alpha}_{(i}a_{j\ell),k} + P^{\alpha}_{mi}\delta^h_{(\alpha}a_{j\ell),k} + P^{\alpha}_{mj}\delta^h_{(i}a_{\alpha\ell),k} + P^{\alpha}_{mk}\delta^h_{(i}a_{j\ell),\alpha} - P^{\alpha}_{ml}\delta^h_{(i}a_{j\alpha),k}\\
& - \theta^h_{ijk\ell,m} + P^h_{\alpha m}\theta^{\alpha}_{ijk\ell} - P^{\alpha}_{mi}\theta^h_{\alpha jk\ell} - P^{\alpha}_{mj}\theta^h_{i\alpha k\ell} - P^{\alpha}_{mk}\theta^h_{ij\alpha\ell} - P^{\alpha}_{m\ell}\theta^h_{ijk\alpha}.\end{aligned}$$

Alternating (13) in ℓ, m we obtain

$$\begin{aligned}\bar{R}^h_{(ij)m;\ell k} - \bar{R}^h_{(ij)\ell;mk} = {} & \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk[lm]}\\
& + \bar{R}^h_{(i|\alpha k|}\bar{R}^{\alpha}_{j)m\ell} + \bar{R}^h_{(ij)\alpha}\bar{R}^{\alpha}_{km\ell} - \bar{R}^{\alpha}_{(ij)k}\bar{R}^h_{\alpha m\ell} + \bar{R}^h_{\alpha(i|k|}\bar{R}^{\alpha}_{j)m\ell}\\
& + \delta^h_{(\alpha}a_{jk)}R^{\alpha}_{i\ell m} + \delta^h_{(\alpha}a_{ik)}R^{\alpha}_{j\ell m} + \delta^h_{(i}a_{j\alpha)}R^{\alpha}_{k\ell m} - \delta^h_{(i}a_{jk)}R^{\alpha}_{\alpha\ell m}\,.\end{aligned} \tag{14}$$

Due to the properties of the Riemannian tensor, (14) can be written as

$$\bar{R}^h_{im\ell;jk} + \bar{R}^h_{jm\ell;ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - N^h_{ijk\ell m}\,, \tag{15}$$

where

$$\begin{aligned}N^h_{ijk\ell m} = {} & T^h_{ijk[\ell m]} + \bar{R}^{\alpha}_{im\ell}\bar{R}^h_{(\alpha j)k} + \bar{R}^{\alpha}_{jm\ell}\bar{R}^h_{(\alpha i)k} + \bar{R}^{\alpha}_{km\ell}\bar{R}^h_{(ij)\alpha}\\
& - \bar{R}^h_{\alpha m\ell}\bar{R}^{\alpha}_{(ij)k} + \delta^h_{(\alpha}a_{jk)}R^{\alpha}_{i\ell m} + \delta^h_{(\alpha}a_{ik)}R^{\alpha}_{j\ell m} + \delta^h_{(\alpha}a_{ij)}R^{\alpha}_{k\ell m} - a_{(ij}R^h_{k)\ell m}.\end{aligned}$$

Let us alternate (15) in j, k. We get

$$\begin{aligned}\bar{R}^h_{jm\ell;ik} - \bar{R}^h_{km\ell;ij} = {} & \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(i}a_{km),j\ell}\\
& - N^h_{i[jk]\ell m} + \bar{R}^h_{\alpha m\ell}\bar{R}^{\alpha}_{ikj} + \bar{R}^h_{i\alpha\ell}\bar{R}^{\alpha}_{mkj} + \bar{R}^h_{im\alpha}\bar{R}^{\alpha}_{\ell kj} - \bar{R}^{\alpha}_{im\ell}\bar{R}^h_{\alpha kj}\,.\end{aligned} \tag{16}$$

Let us interchange i and k in (15), and then use (16). We evaluate

$$\begin{aligned} 2\bar{R}^h_{jm\ell;ik} &= \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} \\ &\quad + \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(j\ell}a_{k),im} + \Omega^h_{ijk\ell m}, \end{aligned} \tag{17}$$

where

$$\Omega^h_{ijk\ell m} = -N^h_{ijk\ell m} + N^h_{k[ij]k\ell m} - \bar{R}^h_{\alpha m\ell}\bar{R}^\alpha_{(kj)i} + \bar{R}^h_{j\alpha\ell}\bar{R}^\alpha_{mik} + \bar{R}^h_{jm\alpha}\bar{R}^\alpha_{\ell ik}$$
$$- \bar{R}^h_{\alpha i(j}\bar{R}^\alpha_{k)m\ell} + \bar{R}^h_{j\alpha\ell}\bar{R}^\alpha_{mik} + \bar{R}^h_{jm\alpha}\bar{R}^\alpha_{\ell ik} - \bar{R}^h_{\alpha m\ell}\bar{R}^\alpha_{ikj} - \bar{R}^h_{i\alpha\ell}\bar{R}^\alpha_{mkj} + \bar{R}^\alpha_{im[\ell}\bar{R}^h_{\alpha]kj}.$$

On the left hand side of (17), let us pass from covariant derivative with respect to $\bar{\nabla}$ to ∇:

$$\begin{aligned} 2\bar{R}^h_{jm\ell,ik} &= \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} \\ &\quad + \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} - \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} S^h_{ijk\ell m} = \Omega^h_{ijk\ell m} - 2\,[&\bar{R}^\alpha_{jm\ell,i}P^h_{\ell k} - \bar{R}^h_{\alpha m\ell,i}P^\alpha_{jk} \\ &- \bar{R}^h_{j\alpha\ell,i}P^\alpha_{mk} - \bar{R}^h_{jm\alpha,i}P^\alpha_{\ell k} - \bar{R}^h_{jm\ell,\alpha}P^\alpha_{ik} \\ &- (\bar{R}^\alpha_{jm\ell}P^\beta_{\alpha i} - \bar{R}^h_{\alpha m\ell}P^\alpha_{ij} - \bar{R}^h_{j\alpha\ell}P^\alpha_{im} - \bar{R}^h_{jm\alpha}P^\alpha_{i\ell})P^h_{\underline{\mathrm{k}}} \\ &- (\bar{R}^\alpha_{jm\ell}P^h_{\alpha\beta} - \bar{R}^h_{\alpha m\ell}P^\alpha_{\beta j} - \bar{R}^h_{j\alpha\ell}P^\alpha_{\beta m} - \bar{R}^h_{jm\alpha}P^\alpha_{\beta\ell})P^\beta_{ik} \\ &- (\bar{R}^\alpha_{\underline{\mathrm{m}}\ell}P^h_{\alpha i} - \bar{R}^h_{\alpha m\ell}P^\alpha_{\underline{\mathrm{i}}} - \bar{R}^h_{\beta\alpha\ell}P^\alpha_{im} - \bar{R}^h_{\beta m\alpha}P^\alpha_{i\ell})P^\beta_{jk} \\ &- (\bar{R}^\alpha_{j\beta\ell}P^h_{\alpha i} - \bar{R}^h_{\alpha\beta\ell}P^\alpha_{ji} - \bar{R}^h_{j\alpha\ell}P^\alpha_{\underline{\mathrm{i}}} - \bar{R}^h_{j\beta\alpha}P^\alpha_{i\ell})P^\beta_{km} \\ &- (\bar{R}^\alpha_{jm\beta}P^h_{\alpha i} - \bar{R}^h_{\alpha m\beta}P^\alpha_{ji} - \bar{R}^h_{j\alpha\beta}P^\alpha_{mi} - \bar{R}^h_{jm\alpha}P^\alpha_{\underline{\mathrm{i}}})P^\beta_{k\ell}]. \end{aligned}$$

Denoting $R^h_{jm\ell i} = \bar{R}^h_{jm\ell,i}$, i. e. introducing a new tensor field of type $(1,4)$ we can write the system (18) in the following form

$$\bar{R}^h_{jm\ell,i} = R^h_{jm\ell i} \tag{19}$$

and

$$\begin{aligned} 2R^h_{jm\ell i,k} &= \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} \\ &\quad + \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}, \end{aligned} \tag{20}$$

where we used (12).

It can be verified that the equations (8), (12), (19) and (20) for functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\bar{R}^h_{ijk}(x)$ and $R^h_{ijkm}(x)$ on (M,∇) form an integrable system; the above functions must satisfy also additional algebraic conditions

$$\begin{gathered} P^h_{ij}(x) = P^h_{ji}(x), \quad a_{ij}(x) = a_{ji}(x), \quad \bar{R}^h_{i(jk)}(x) = \bar{R}^h_{(ijk)}(x) = 0, \\ R^h_{i(jk)\ell}(x) = R^h_{(ijk)\ell}(x) = 0. \end{gathered} \tag{21}$$

So we have succeeded to prove the following generalization of the result of Sinyukov [7, 8] (we use the above notation).

Theorem. *Let (M, ∇) be a manifold with affine connection and $(\bar{M}, \bar{\nabla})$ a generalized Ricci-symmetric manifold. There is a $\tilde{\pi}_1$ mapping $f\colon M \to \bar{M}$ (i.e. a canonical almost geodesic mapping of type π_1) if and only if there exist functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\bar{R}^h_{ijk}(x)$ and $R^h_{ijkm}(x)$ which satisfy the equations (8), (12), (19), (20), and (21). The system of equations (8), (12), (19) and (20) forms a Cauchy type system of PDE's in covariant derivatives.*

As a consequence we obtain

Corollary. *The family of all generalized Ricci-symmetric manifolds, which can serve as an image of the given affine manifold (M, ∇) under some $\tilde{\pi}_1$-mapping, depends on at most*

$$\frac{1}{6}\, n(n+1)(2n^3 - 4n^2 + 5n + 3) \tag{22}$$

parameters.

References

[1] V. Berezovski and J. Mikeš. On a classification of almost geodesic mappings of affine connection spaces. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, 35:21–24 (1997), 1996.

[2] V. E. Berezovski, J. Mikeš, and A. Vanžurová. Canonical almost geodesic mappings of type $\tilde{\pi}_1$ onto pseudo-Riemannian manifolds. In *Differential geometry and its applications*, pages 65–75. World Sci. Publ., Hackensack, NJ, 2008.

[3] V. M. Chernyshenko. Affine-connected spaces with a correspondent complex of geodesics. *Collection of Works of Mech.-Math. Chair of Dnepropetrovsk Univ.*, 6:105–118, 1961.

[4] J. Mikeš, V. Kiosak, and A. Vanžurová. *Geodesic mappings of manifolds with affine connection.* Palacký University Olomouc, Olomouc, 2008.

[5] N. S. Sinyukov. Almost geodesic mappings of affinely connected and Riemannian spaces. *Dokl. Akad. Nauk SSSR*, 151:781–782, 1963.

[6] N. S. Sinyukov. *Geodezicheskie otobrazheniya rimanovykh prostranstv.* "Nauka", Moscow, 1979.

[7] N. S. Sinyukov. Almost geodesic mappings of affine connected and Riemannian spaces. *Itogi Nauki Tekh., Ser. Probl. Geom.*, 13:3–26, 1982.

[8] N. S. Sinyukov. Almost-geodesic mappings of affinely connected and Riemann spaces. *J. Sov. Math.*, 25:1235–1249, 1984.

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