

ON S-3 LIKE FOUR-DIMENSIONAL FINSLER SPACES

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ABSTRACT. In 1977, M. Matsumoto and R. Miron [9] constructed an orthonormal frame for an n -dimensional Finsler space, called ‘Miron frame’. The present authors [1, 2, 3, 10, 11] discussed four-dimensional Finsler spaces equipped with such frame. M. Matsumoto [7, 8] proved that in a three-dimensional Berwald space, all the main scalars are h -covariant constants and the h -connection vector vanishes. He also proved that in a three-dimensional Finsler space satisfying T-condition, all the main scalars are functions of position only and the v -connection vector vanishes [6, 7]. The purpose of the present paper is to generalize these results for an S-3 like four-dimensional Finsler space.

1. PRELIMINARIES

Let M^4 be a four-dimensional smooth manifold and $F^4 = (M^4, L)$ be a four-dimensional Finsler space equipped with a metric function $L(x, y)$ on M^4 . The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by $l_i = \dot{\partial}_i L$, $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$, $h_{ij} = L \dot{\partial}_i \dot{\partial}_j L$ and $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$ respectively. The torsion vector C^i is defined by $C^i = C_{jk}^i g^{jk}$. Throughout this paper, we use the symbols $\dot{\partial}_i$ and ∂_i for $\partial/\partial y^i$ and $\partial/\partial x^i$ respectively. The Cartan connection in the Finsler space is given as $CT = (F_{jk}^i, G_j^i, C_{jk}^i)$. The h - and v -covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are given by

$$(1.1) \quad X_{i|j} = \partial_j X_i - (\dot{\partial}_h X_i) G_j^h - F_{ij}^r X_r,$$

and

$$(1.2) \quad X_i|_j = \dot{\partial}_j X_i - C_{ij}^r X_r.$$

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The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors $(e_1^i, e_2^i, e_3^i, e_4^i)$. The first vector e_1^i is the normalized supporting element l^i and the second e_2^i is the normalized torsion vector $m^i = C^i/\tilde{c}$, where $\tilde{c}$ is the length of the torsion vector C^i . The third $e_3^i = n^i$ and the fourth $e_4^i = p^i$ are constructed by the method of Matsumoto and Miron [9]. With respect to this frame, the scalar components of an arbitrary tensor T_j^i are defined by

$$(1.3) \quad T_{\alpha\beta} = T_j^i e_{\alpha i} e_{\beta}^j.$$

From this, we get

$$(1.4) \quad T_j^i = T_{\alpha\beta} e_{\alpha}^i e_{\beta j},$$

where summation convention is also applied to Greek indices. The scalar components of the metric tensor g_{ij} are $\delta_{\alpha\beta}$. Therefore we get

$$(1.5) \quad g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j.$$

Let $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}/L$ be scalar components of the h - and v -covariant derivatives $e_{\alpha i|j}^i$ and $e_{\alpha i}^i|_j$ respectively of the vectors e_{α}^i , then

$$(1.6) \quad e_{\alpha i|j}^i = H_{\alpha)\beta\gamma} e_{\beta}^i e_{\gamma j},$$

and

$$(1.7) \quad L e_{\alpha}^i|_j = V_{\alpha)\beta\gamma} e_{\beta}^i e_{\gamma j}.$$

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called h - and v -connection scalars respectively and are positively homogeneous of degree 0 in y .

Orthogonality of the Miron frame yields

$$H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma} \text{ and } V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}.$$

Also we have $H_{1)\beta\gamma} = 0$ and $V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}$ [7].

Now we define Finsler vector fields :

$$h_i = H_{2)3\gamma} e_{\gamma i}, \quad j_i = H_{4)2\gamma} e_{\gamma i}, \quad k_i = H_{3)4\gamma} e_{\gamma i},$$

and

$$u_i = V_{2)3\gamma} e_{\gamma i}, \quad v_i = V_{4)2\gamma} e_{\gamma i}, \quad w_i = V_{3)4\gamma} e_{\gamma i}.$$

The vector fields h_i, j_i, k_i are called h -connection vectors and the vector fields u_i, v_i, w_i are called v -connection vectors. The scalars $H_{2)3\gamma}, H_{4)2\gamma}, H_{3)4\gamma}$ and $V_{2)3\gamma}, V_{4)2\gamma}, V_{3)4\gamma}$ are considered as the scalar components $h_{\gamma}, j_{\gamma}, k_{\gamma}$ and $u_{\gamma}, v_{\gamma}, w_{\gamma}$ of the h - and v -connection vectors respectively with respect to the orthonormal frame.

Let $C_{\alpha\beta\gamma}$ are the scalar components of LC_{ijk} then

$$(1.8) \quad LC_{ijk} = C_{\alpha\beta\gamma} e_{\alpha i} e_{\beta j} e_{\gamma k}.$$

The main scalars of a four-dimensional Finsler space are given by [1, 3, 11]

$$\begin{aligned} C_{222} &= A, \quad C_{233} = B, \quad C_{244} = C, \quad C_{322} = D, \\ C_{333} &= E, \quad C_{422} = F, \quad C_{433} = G, \quad C_{234} = H. \end{aligned}$$

We also have $C_{344} = -(D + E)$, $C_{444} = -(F + G)$ and

$$(1.9) \quad A + B + C = L\tilde{C}.$$

$L\tilde{C}$ is called the unified main scalar.

Taking h -covariant differentiation of (1.4), we get

$$(1.10) \quad T_{j|k}^i = (\delta_k T_{\alpha\beta}) e_{\alpha}^i e_{\beta}^j + T_{\alpha\beta} e_{\alpha}^i e_{\beta}^j + T_{\alpha\beta} e_{\alpha}^i e_{\beta}^j e_{\gamma}^k,$$

where $\delta_k = \partial_k - G_k^r \dot{\partial}_r$. If $T_{\alpha\beta,\gamma}$ are scalar components of $T_{j|k}^i$, i.e.

$$(1.11) \quad T_{j|k}^i = T_{\alpha\beta,\gamma} e_{\alpha}^i e_{\beta}^j e_{\gamma}^k,$$

then we obtain

$$(1.12) \quad T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta}) e_{\gamma}^k + T_{\mu\beta} H_{\mu\alpha\gamma} + T_{\alpha\mu} H_{\mu\beta\gamma}.$$

Similarly, if $T_{\alpha\beta;\gamma}$ are scalar components of $LT_j^i|_k$, i.e.

$$(1.13) \quad LT_j^i|_k = T_{\alpha\beta;\gamma} e_{\alpha}^i e_{\beta}^j e_{\gamma}^k,$$

then we get

$$(1.14) \quad T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta}) e_{\gamma}^k + T_{\mu\beta} V_{\mu\alpha\gamma} + T_{\alpha\mu} V_{\mu\beta\gamma}.$$

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are respectively called h - and v -scalar derivatives of scalar components $T_{\alpha\beta}$ of T .

2. T -CONDITION

The tensor T_{hijk} defined by

$$(2.1) \quad T_{hijk} = LC_{hij}|_k + C_{hij}l_k + C_{hik}l_j + C_{hjk}l_i + C_{kij}l_h,$$

is called T -tensor in a Finsler space. It is completely symmetric in its indices. A Finsler space is said to satisfy T -condition if the T -tensor T_{hijk} vanishes identically.

We are concerned with the tensor $C_{hij}|_k$. From (1.8) and (1.13), it follows that

$$L^2 C_{hij}|_k + LC_{hij}l_k = C_{\alpha\beta\gamma;\delta} e_{\alpha}^h e_{\beta}^i e_{\gamma}^j e_{\delta}^k,$$

which implies

$$(2.2) \quad L^2 C_{hij}|_k = (C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta}) e_{\alpha}^h e_{\beta}^i e_{\gamma}^j e_{\delta}^k.$$

Therefore the scalar components $T_{\alpha\beta\gamma\delta}$ of LT_{hijk} are given by

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta} + \delta_{1\alpha} C_{\beta\gamma\delta} + \delta_{1\beta} C_{\alpha\gamma\delta} + \delta_{1\gamma} C_{\alpha\beta\delta}.$$

From $T_{hijk}l^k = 0$, we have $T_{\alpha\beta\gamma 1} = 0$. Thus the surviving components $T_{\alpha\beta\gamma\delta}$ are only

$$(2.3) \quad T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta}; \quad \alpha, \beta, \gamma, \delta = 2, 3, 4.$$

Using (1.14), the explicit forms of $C_{\alpha\beta\gamma;\delta}$ are obtained as follows:

$$(2.4) \quad \left\{ \begin{array}{l} a) C_{222;\delta} = A_{;\delta} - 3Du_{\delta} + 3Fv_{\delta}, \\ b) C_{233;\delta} = B_{;\delta} + (2D - E)u_{\delta} + Gv_{\delta} - 2Hw_{\delta}, \\ c) C_{244;\delta} = C_{;\delta} + (D + E)u_{\delta} - (3F + G)v_{\delta} + 2Hw_{\delta}, \\ d) C_{322;\delta} = D_{;\delta} + (A - 2B)u_{\delta} + 2Hv_{\delta} - Fw_{\delta}, \\ e) C_{333;\delta} = E_{;\delta} + 3Bu_{\delta} - 3Gw_{\delta}, \\ f) C_{422;\delta} = F_{;\delta} - 2Hu_{\delta} - (A - 2C)v_{\delta} + Dw_{\delta}, \\ g) C_{433;\delta} = G_{;\delta} + 2Hu_{\delta} - Bv_{\delta} + (2D + 3E)w_{\delta}, \\ h) C_{234;\delta} = H_{;\delta} + (F - G)u_{\delta} - (2D + 3E)v_{\delta} + (B - C)w_{\delta}, \\ i) C_{344;\delta} = -D_{;\delta} - E_{;\delta} + Cu_{\delta} - 2Hv_{\delta} + (F + 3G)w_{\delta}, \\ j) C_{444;\delta} = -F_{;\delta} - G_{;\delta} - 3Cv_{\delta} - (3D + 3E)w_{\delta}, \\ k) C_{1\beta\gamma;\delta} = -C_{\beta\gamma\delta}, \end{array} \right.$$

where $A_{;\delta} = L(\dot{\partial}_k A)e_{\delta}^k$. From (1.9) and (2.4), we get

$$(2.5) \quad \left\{ \begin{array}{l} C_{222;\delta} + C_{233;\delta} + C_{244;\delta} = A_{;\delta} + B_{;\delta} + C_{;\delta} = (A + B + C)_{;\delta} = (L\tilde{C})_{;\delta}, \\ C_{322;\delta} + C_{333;\delta} + C_{344;\delta} = L\tilde{C}u_{\delta}, \\ C_{422;\delta} + C_{433;\delta} + C_{444;\delta} = -L\tilde{C}v_{\delta}. \end{array} \right.$$

Thus from (2.3), (2.4) and (2.5), we have

Theorem 2.1. *In a four-dimensional Finsler space satisfying T-condition, the v-connection vectors u_i and v_i vanish identically. Also main scalar A and the unified main scalar $L\tilde{C}$ are v-covariant constants (functions of position only). Furthermore, if v-connection vector w_i vanishes then all the main scalars are functions of position only.*

3. BERWALD SPACE

A Berwald space is characterized by $C_{hij|k} = 0$. From (1.8) and (1.11), it follows that

$$(3.1) \quad LC_{hij|k} = C_{\alpha\beta\gamma,\delta}e_{\alpha)h}e_{\beta)i}e_{\gamma)j}e_{\delta)k},$$

where $C_{\alpha\beta\gamma,\delta}$ are given by

$$C_{\alpha\beta\gamma,\delta} = (\delta_k C_{\alpha\beta\gamma})e_{\delta}^k + C_{\mu\beta\gamma}H_{\mu)\alpha\delta} + C_{\alpha\mu\gamma}H_{\mu)\beta\delta} + C_{\alpha\beta\mu}H_{\mu)\gamma\delta}.$$

The explicit forms of $C_{\alpha\beta\gamma,\delta}$ are obtained as follows:

$$(3.2) \quad \left\{ \begin{array}{l} a) C_{222,\delta} = A_{,\delta} - 3Dh_{\delta} + 3Fj_{\delta}, \\ b) C_{233,\delta} = B_{,\delta} + (2D - E)h_{\delta} + Gj_{\delta} - 2Hk_{\delta}, \\ c) C_{244,\delta} = C_{,\delta} + (D + E)h_{\delta} - (3F + G)j_{\delta} + 2Hk_{\delta}, \\ d) C_{322,\delta} = D_{,\delta} + (A - 2B)h_{\delta} + 2Hj_{\delta} - Fk_{\delta}, \\ e) C_{333,\delta} = E_{,\delta} + 3Bh_{\delta} - 3Gk_{\delta}, \\ f) C_{422,\delta} = F_{,\delta} - 2Hh_{\delta} - (A - 2C)j_{\delta} + Dk_{\delta}, \\ g) C_{433,\delta} = G_{,\delta} + 2Hh_{\delta} - Bj_{\delta} + (2D + 3E)k_{\delta}, \\ h) C_{234,\delta} = H_{,\delta} + (F - G)h_{\delta} - (2D + 3E)j_{\delta} + (B - C)k_{\delta}, \\ i) C_{344,\delta} = -D_{,\delta} - E_{,\delta} + Ch_{\delta} - 2Hj_{\delta} + (F + 3G)k_{\delta}, \\ j) C_{444,\delta} = -F_{,\delta} - G_{,\delta} - 3Cj_{\delta} - (3D + 3E)k_{\delta}, \\ k) C_{1\beta\gamma,\delta} = 0. \end{array} \right.$$

From (1.9) and (3.2), we get

$$(3.3) \quad \begin{aligned} C_{322,\delta} + C_{333,\delta} + C_{344,\delta} &= (A + B + C)h_{\delta} = L\tilde{c}h_{\delta}, \\ C_{422,\delta} + C_{433,\delta} + C_{444,\delta} &= -(A + B + C)j_{\delta} = -L\tilde{c}j_{\delta}, \\ C_{222,\delta} + C_{233,\delta} + C_{244,\delta} &= (A_{,\delta} + B_{,\delta} + C_{,\delta}) = (A + B + C)_{,\delta}. \end{aligned}$$

Thus from (3.2) and (3.3), we have:

Theorem 3.1 ([11]). *In a four-dimensional Berwald space, the h-connection vectors h_i and j_i vanish identically. Also main scalar A and the unified main scalar $L\tilde{c}$ are h-covariant constants. Furthermore, if h-connection vector k_i vanishes then all the main scalars are h-covariant constants.*

4. v-CURVATURE TENSOR

The v-curvature tensor is defined by

$$(4.1) \quad S_{hijk} = C_{hk}^r C_{ijr} - C_{hj}^r C_{ikr}.$$

The scalar components $S_{\alpha\beta\gamma\delta}$ of $L^2 S_{hijk}$ are given by

$$(4.2) \quad L^2 S_{hijk} = S_{\alpha\beta\gamma\delta} e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}.$$

Since S_{hijk} is skew-symmetric in h and i as well as j and k and $S_{0ijk} = S_{hi0k} = 0$, the surviving independent components of $S_{\alpha\beta\gamma\delta}$ are only six, which are given by

$$\begin{aligned} S_{2323} &= C_{23\mu}C_{\mu32} - C_{22\mu}C_{\mu33} = D^2 + B^2 + H^2 - AB - DE - FG, \\ S_{2424} &= C_{24\mu}C_{\mu42} - C_{22\mu}C_{\mu44} = 2F^2 + H^2 + C^2 + D^2 - AC + DE + FG, \\ S_{3434} &= C_{34\mu}C_{\mu34} - C_{33\mu}C_{\mu44} = H^2 + 2G^2 + D^2 + 2E^2 + 3DE - BC + FG, \\ S_{2334} &= C_{24\mu}C_{\mu33} - C_{23\mu}C_{\mu34} = BF + 2EH + CG - BG, \\ S_{2434} &= C_{24\mu}C_{\mu34} - C_{23\mu}C_{\mu44} = 2FH + 2GH - 2CD - CE + BD + BE, \\ S_{2324} &= C_{24\mu}C_{\mu23} - C_{22\mu}C_{\mu34} = 2FD + BH + CH - AH - DG + EF. \end{aligned}$$

A Finsler space F^n ($n \geq 4$) is called S-3 like, if there exists a scalar S such that the curvature tensor S_{hijk} of F^n is written in the form

$$(4.3) \quad L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}).$$

Let us consider a four-dimensional S-3 like Finsler space. Then

$$\begin{aligned} L^2 S_{hijk} &= S(h_{hj}h_{ik} - h_{hk}h_{ij}) \\ &= S[(m_h m_j + n_h n_j + p_h p_j)(m_i m_k + n_i n_k + p_i p_k) \\ &\quad - (m_h m_k + n_h n_k + p_h p_k)(m_i m_j + n_i n_j + p_i p_j)] \\ &= S[(m_h n_i - m_i n_h)(m_j n_k - m_k n_j) + (m_h p_i - m_i p_h)(m_j p_k - m_k p_j) \\ &\quad + (n_h p_i - n_i p_h)(n_j p_k - n_k p_j)]. \end{aligned}$$

This implies that the scalar components are

$$S_{2323} = S, \quad S_{2324} = 0, \quad S_{2334} = 0, \quad S_{2424} = S, \quad S_{2434} = 0, \quad S_{3434} = S.$$

M. Matsumoto [5] proved that the v-curvature S of an S-3 like Finsler space is function of position only. Therefore in S-3 like four-dimensional Finsler space, six functions $D^2 + B^2 + H^2 - AB - DE - FG$, $2F^2 + H^2 + C^2 + D^2 - AC + DE + FG$, $H^2 + 2G^2 + D^2 + 2E^2 + 3DE - BC + FG$, $BF + 2EH + CG - BG$, $2FH + 2GH - 2CD - CE + BD + BE$ and $2FD + BH + CH - AH - DG + EF$ are functions of position only. In view of theorem 2.1 and equation (1.9), functions A and $A + B + C$ are functions of position only in a four-dimensional Finsler space satisfying T -condition. Thus, in an S-3 like Finsler space satisfying T -condition, eight functions A , $A + B + C$, $D^2 + B^2 + H^2 - AB - DE - FG$, $2F^2 + H^2 + C^2 + D^2 - AC + DE + FG$, $H^2 + 2G^2 + D^2 + 2E^2 + 3DE - BC + FG$, $BF + 2EH + CG - BG$, $2FH + 2GH - 2CD - CE + BD + BE$ and $2FD + BH + CH - AH - DG + EF$ are functions of position only. These eight functions are clearly independent and therefore the main scalars A, B, C, D, E, F, G and H are functions of position only. Thus, we have:

Theorem 4.1. *In an S-3 like four-dimensional Finsler space satisfying T -condition, all the main scalars are functions of position only.*

It is clear from (2.4) that if all the main scalars are functions of position only in a Finsler space satisfying T -condition, then the v -connection vectors u_i, v_i , and w_i vanish. This leads to:

Theorem 4.2. *In an S-3 like four-dimensional Finsler space satisfying T -condition, the v -connection vectors u_i, v_i , and w_i vanish identically.*

A Landsberg space is characterized by $C_{hij|k} = C_{hik|j}$. H. Yasuda [12] proved that in an S-3 like Landsberg space, the v -curvature S is constant. In view of this result, in an S-3 like four-dimensional Landsberg space, six independent functions $D^2 + B^2 + H^2 - AB - DE - FG$, $2F^2 + H^2 + C^2 + D^2 - AC + DE + FG$, $H^2 + 2G^2 + D^2 + 2E^2 + 3DE - BC + FG$, $BF + 2EH + CG - BG$, $2FH + 2GH - 2CD - CE + BD + BE$ and $2FD + BH + CH - AH - DG + EF$ are constants. Since every Berwald space is a Landsberg space, these six functions are constant in an S-3 like Berwald space. From theorem 3.1 and equation (1.9), functions A and $A + B + C$ are h -covariant constants in a four-dimensional Berwald space. Therefore in an S-3 like Berwald space, eight independent functions $A, A + B + C, D^2 + B^2 + H^2 - AB - DE - FG, 2F^2 + H^2 + C^2 + D^2 - AC + DE + FG, H^2 + 2G^2 + D^2 + 2E^2 + 3DE - BC + FG, BF + 2EH + CG - BG, 2FH + 2GH - 2CD - CE + BD + BE$ and $2FD + BH + CH - AH - DG + EF$ are h -covariant constants and therefore the main scalars A, B, C, D, E, F, G and H are h -covariant constants.

Thus, we have:

Theorem 4.3. *In an S-3 like four-dimensional Berwald space, all the main scalars are h -covariant constants.*

It is clear from (3.2) that if all the main scalars are h -covariant constants in a Berwald space, then the h -connection vectors h_i, j_i and k_i vanish.

This leads to:

Theorem 4.4. *In an S-3 like four-dimensional Berwald space, the h -connection vectors h_i, j_i and k_i vanish identically.*

In view of theorems 4.1, 4.2, 4.3 and 4.4, we can say

Theorem 4.5. *In an S-3 like four-dimensional Berwald space satisfying T -condition, all the main scalars are constants and the h - and v -connection vectors vanish.*

F. Ikeda [4] proved that a Landsberg space satisfying T -condition is a Berwald space. Thus, we may conclude:

Theorem 4.6. *In an S-3 like four-dimensional Landsberg space satisfying T -condition, all the main scalars are constants and the h - and v -connection vectors vanish.*

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