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ORE EXTENSIONS OVER NEAR PSEUDO-VALUATION RINGS AND NOETHERIAN RINGS

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Abstract. We recall that a ring R is called near pseudo-valuation ring if every minimal prime ideal is a strongly prime ideal.

Let R be a commutative ring, σ an automorphism of R and δ a σ-derivation of R. We recall that a prime ideal P of R is δ-divided if it is comparable (under inclusion) to every σ-invariant and δ-invariant ideal I (i.e. $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$) of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided. A ring R is said to be almost δ-divided ring if every minimal prime ideal of R is δ-divided.

Recall that an endomorphism σ of a ring R is called Min.Spec-type if $\sigma(U) \subseteq U$ for all minimal prime ideals U of R and R is a Min.Spec-type ring (if there exists a Min.Spec-type endomorphism of R). With this we prove the following.

Let R be a commutative Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$ is the field of rational numbers), σ a Min.Spec-type automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further let any strongly prime ideal U of R with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ implies that $U[x;\sigma,\delta]$ is a strongly prime ideal of $R[x;\sigma,\delta]$. Then

(1) R is a near pseudo valuation ring implies that $R[x;\sigma,\delta]$ is a near pseudo valuation ring
(2) R is an almost δ-divided ring if and only if $R[x;\sigma,\delta]$ is an almost δ-divided ring.

1. Introduction

We follow the notation as in Bhat [14]. All rings are associative with identity. Throughout this paper R denotes a commutative ring with identity $1 \neq 0$. The set of prime ideals of R is denoted by $\mathrm{Spec}(R)$, the set of minimal prime ideals of R is denoted by $\mathrm{Min.\,Spec}(R)$, and the set of strongly prime ideals is denoted

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by $\mathrm{S.\,Spec}(R)$. The fields of rational numbers and real numbers are denoted by $\mathbb{Q}$ and $\mathbb{R}$ respectively unless otherwise stated.

We recall that as in Hedstrom and Houston [16], an integral domain R with quotient field F, is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). For a survey article on pseudo-valuation domains, the reader is referred to Badawi [6]

In Badawi, Anderson and Dobbs [8], the study of pseudo-valuation domains was generalized to arbitrary rings in the following way. A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion; i.e. $aP \subseteq bR$ or $bR \subseteq aP$) for all $a, b \in R$. A ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. For more details on pseudo-valuation rings, the reader is referred to Badawi [7].

The concept of pseudo-valuation domain is generalized to the context of rings with zero divisors as in [8, 1, 3, 4, 5].

This article concerns the study of skew polynomial rings over PVDs. Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R ($\delta\colon R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$). In case σ is identity, δ is just called a derivation. For example let $R = F[x]$, F a field. Then $\sigma\colon R \to R$ defined by $\sigma(f(x)) = f(0)$ is an endomorphism of R. Also let $K = \mathbb{R} \times \mathbb{R}$. Then $g\colon K \to K$ by $g(a, b) = (b, a)$ is an automorphism of K.

Let σ be an automorphism of a ring R and $\delta\colon R \to R$ any map. Let $\phi\colon R \to M_2(R)$ be a map defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R if and only if ϕ is a homomorphism.

Also let $R = F[x]$, F a field. Then the usual differential operator $\frac{d}{dx}$ is a derivation of R.

We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. If I is an ideal of R such that I is σ-invariant; i.e. $\sigma(I) \subseteq I$ and I is δ-invariant; i.e. $\delta(I) \subseteq I$, then we denote $I[x; \sigma, \delta]$ by $O(I)$. We would like to mention that $R[x; \sigma, \delta]$ is the usual set of polynomials with coefficients in R, i.e. $\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$.

In case δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) \subseteq I$, we denote $I[x; \sigma]$ by $S(I)$. In case σ is the identity map, we denote the differential operator ring $R[x; \delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x; \delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example see [10, 11, 12, 13, 14, 15].

Near Pseudo-valuation rings. Recall that a ring R is called a near pseudo-valuation ring (NPVR) if each minimal prime ideal P of R is strongly prime (Bhat [13]). For example a reduced ring is NPVR. Here the term near may not be interpreted as near ring (Bell and Mason [9]). We note that a near pseudo-valuation ring (NPVR) is a pseudo-valuation ring (PVR), but the converse is not true. For example a reduced ring is a NPVR, but need not be a PVR.

Divided rings. We recall that a prime ideal P of R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided (Badawi [2]). It is known (Lemma (1) of Badawi, Anderson and Dobbs [8]) that a pseudo-valuation ring is a divided ring. Recall that a ring R is called an almost divided ring if every minimal prime ideal of R is divided (Bhat [13]).

δ-divided rings. A prime ideal P of R is said to be δ-divided (where δ is a σ-derivation of R) if it is comparable (under inclusion) to every σ-invariant and δ-invariant ideal I of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided (Bhat [11]). A ring R is said to be almost δ-divided ring if every minimal prime ideal of R is δ-divided (Bhat [13]). For more details on near pseudo-valuation rings, δ-divided rings and almost δ-divided rings the reader is referred to [11, 13, 14].

The author of this paper has proved the following in [14] concerning strongly prime ideals of Ore extensions.

Theorem B (Bhat [14])**.** *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let δ be a derivation of R. Further let any $U \in \mathrm{S.Spec}(R)$ with $\delta(U) \subseteq U$ implies that $O(U) \in \mathrm{S.Spec}(O(R)$. Then*

1. *R is a near pseudo-valuation ring implies that $D(R)$ is a near pseudo-valuation ring*
2. *R is an almost δ-divided ring if and only if $D(R)$ is an almost δ-divided ring.*

Theorem BB (Bhat [14])**.** *Let R be a Noetherian ring. Let σ be a Min.Spec-type automorphism of R. Further let any $U \in \mathrm{S.Spec}(R)$ with $\sigma(U) = U$ implies that $O(U) \in \mathrm{S.Spec}(O(R)$. Then*

1. *R is a near pseudo-valuation ring implies that $S(R)$ is a near pseudo-valuation ring*
2. *R is an almost σ-divided ring if and only if $S(R)$ is an almost σ-divided ring.*

In this paper we generalize the above results of [14] and answer the problem posed in [14].

Theorem A. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ a Min.Spec-type automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further let any $U \in \mathrm{S.Spec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ implies that $O(U) \in \mathrm{S.Spec}(O(R))$. Then*

1. *R is a near pseudo valuation ring implies that $R[x; \sigma, \delta]$ is a near pseudo valuation ring*
2. *R is an almost δ-divided ring if and only if $R[x; \sigma, \delta]$ is an almost δ-divided ring.*

This is proved in Theorem (2.5), but before that, we have the following definition.

Definition 1.1 (see [14]). Let R be a ring. We say that an endomorphism σ of R is Min.Spec-type if $\sigma(U) \subseteq U$ for all minimal prime ideals U of R. We say that a ring R is Min.Spec-type ring if there exists a Min.Spec-type endomorphism of R.

Example 1.2 (see [14]). Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Let $\sigma\colon R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is a Min.Spec-type endomorphism of R, and therefore, R is a Min.Spec-type ring.

2. Proof of Main Theorem

Theorem 2.1. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be a Min.Spec-type automorphism of R and δ a σ-derivation of R. Then $\delta(U) \subseteq U$ for all $U \in \operatorname{Min.Spec}(R)$.*

Proof. Let $U \in \operatorname{Min.Spec}(R)$. We have $\sigma(U) \subseteq U$. Consider the set

$$T = \{a \in U \mid \text{ such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}.$$

First of all, we will show that T is an ideal of R. Let $a, b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1\}$. Therefore $a - b \in T$. Therefore T is a δ-invariant ideal of R.

We will now show that $T \in \operatorname{Spec}(R)$. Suppose $T \notin \operatorname{Spec}(R)$. Let $a \notin T$, $b \notin T$ be such that $aRb \subseteq T$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq T$. This implies on simplification that

$$\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U,$$

where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $T \in \operatorname{Spec}(R)$. Now $T \subseteq U$, so $T = U$ as $U \in \operatorname{Min.Spec}(R)$. Hence, $\delta(U) \subseteq U$. ☐

Lemma 2.2. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be a Min.Spec-type automorphism of R and δ a σ-derivation of R. Then*

1. *if U is a minimal prime ideal of R, then $O(U)$ is a minimal prime ideal of of $O(R)$ and $O(U) \cap R = U$*
2. *if P is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of R.*

Proof. (1) Let U be a minimal prime ideal of R. Now $\sigma(U) \subseteq U$ and by Theorem (2.1) $\delta(U) \subseteq U$. Now, on the same lines as in Theorem (2.22) of

Goodearl and Warfield [15] we have $O(U) \in \mathrm{Spec}(O(R))$. Suppose $L \subset O(U)$ be a minimal prime ideal of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of R, a contradiction. Therefore $O(U) \in \mathrm{Min.Spec}(O(R))$. Now it is easy to see that $O(U) \cap R = U$.

(2) We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Now, the proof follows on the same lines as in Theorem (2.22) of Goodearl and Warfield [15] using Lemma (2.1) and Lemma (2.2) of Bhat [11] and Theorem (2.1). □

Theorem 2.3. *Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then the ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.*

Proof. See Theorem (1.12) of Goodearl and Warfield [15]. □

Remark 2.4. Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x; \sigma, \delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x; \sigma, \delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.

We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$ for all $a \in R$, then the above does not hold. For example let $f(x) = xa$ and $g(x) = xb$, $a, b \in R$. Then

$$\overline{\delta}(f(x)g(x)) = x^2\{\delta(\sigma(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\},$$

but

$$\begin{aligned} &\overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x)) = \\ &\qquad = x^2\{\sigma(\delta(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\}. \end{aligned}$$

We are now in a position to prove Theorem A as follows.

Theorem 2.5. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ a Min.Spec-type automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further let any $U \in \mathrm{S.Spec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ implies that $O(U) \in \mathrm{S.Spec}(O(R)$. Then*

1. *R is a near pseudo valuation ring implies that $R[x; \sigma, \delta]$ is a near pseudo valuation ring*
2. *R is an almost δ-divided ring if and only if $R[x; \sigma, \delta]$ is an almost δ-divided ring.*

Proof. (1) Let R be a near pseudo-valuation ring which is also an algebra over $\mathbb{Q}$. Now $O(R)$ is Noetherian by Theorem (2.3). Let $J \in \mathrm{Min.Spec}(O(R))$. Then by Lemma (2.2) $J \cap R \in \mathrm{Min.Spec}(R)$. Now R is a near pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \in \mathrm{S.Spec}(R)$. Also $\delta(J \cap R) \subseteq J \cap R$ by Theorem (2.1). Now Lemma (2.2) implies that $O(J \cap R) = J$, and by hypothesis $O(J \cap R) \in \mathrm{S.Spec}(O(R))$. Therefore, $J \in \mathrm{S.Spec}(O(R))$. Hence $O(R)$ is a near pseudo-valuation ring.

(2) Let R be an almost δ-divided which is also an algebra over $\mathbb{Q}$. Now $O(R)$ is Noetherian by Theorem (2.3). Let $J \in \mathrm{Min.Spec}(O(R))$ and K be an ideal

of $O(R)$ such that $\sigma(K) \subseteq K$ and $\delta(K) \subseteq K$. Note that σ can be extended to an automorphism of $O(R)$ and δ can be extended to a σ-derivation of $O(R)$ by Remark (2.4). Now by Lemma (2.2) $J \cap R \in \mathrm{Min.Spec}(R)$. Now R is an almost δ-divided commutative Noetherian $\mathbb{Q}$-algebra, therefore $J \cap R$ and $K \cap R$ are comparable (under inclusion), say $J \cap R \subseteq K \cap R$. Now $\delta(K \cap R) \subseteq K \cap R$. Therefore, $O(K \cap R)$ is an ideal of $O(R)$ and so $O(J \cap R) \subseteq O(K \cap R)$. This implies that $J \subseteq K$. Hence $O(R)$ is an almost δ-divided ring.

Conversely suppose that $O(R)$ is almost δ-divided. Let $U \in \mathrm{Min.Spec}(R)$ and V be an ideal of R such that $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$. Now by Theorem (2.1) $\delta(U) \subseteq U$, and Lemma (2.2) implies that $O(U) \in \mathrm{Min.Spec}(O(R))$. Now $O(R)$ is an almost δ-divided ring, therefore $O(U)$ and $O(V)$ are comparable (under inclusion), say $O(U) \subseteq O(V)$. Therefore, $O(U) \cap R \subseteq O(V) \cap R$; i.e. $U \subseteq V$. Hence R is an almost δ-divided ring. □

We note that in above Theorem the hypothesis that any $U \in \mathrm{S.Spec}(R)$ with $\delta(U) \subseteq U$ implies that $O(U) \in \mathrm{S.Spec}(O(R))$ can not be deleted as extension of a strongly prime ideal of R need not be a strongly prime ideal of $O(R)$.

Example 2.6 (see [14]). $R = \mathbb{Z}_{(p)}$. This is in fact a discrete valuation domain, and therefore, its maximal ideal $P = pR$ is strongly prime. But, $pR[x]$ is not strongly prime in $R[x]$ because it is not comparable with $xR[x]$ (so the condition of being strongly prime in $R[x]$ fails for $a = 1$ and $b = x$).

Question 2.7. Let R be a NPVR. Let σ be an automorphism of R and δ a σ-derivation of R. Is $O(R) = R[x; \sigma, \delta]$ a NPVR?

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