

ON A MULTIPLIER OF THE PROGRESSIVE MEANS AND CONVEX MAPS OF THE UNIT DISC

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ABSTRACT. In this paper we are concerned with a multiplier $\varpi(n)$ of the Progressive means, and convex maps of the unit disc. With this concern we would have brought up in a rather unified approach the results of G. Pólya and I. J. Schoenberg in [7], T. Başgöze, J. L. Frank, and F. R. Keogh in [3], and Ziad S. Ali in [1]. More theorems on the properties of the multiplier $\varpi(n)$ are given, and a key lemma showing combinatorial trigonometric identities whose offsprings are: Several combinatorial, and combinatorial trigonometric identities, and a new method for generating the Chebyshev's polynomials. Finally we present a different form of $\varpi(n)$ as well as relating $\varpi(n)$ to the subordination principle.

1. INTRODUCTION

Let $\sum_{k=0}^{\infty} u_k$ be a given series, and let $\{S_n\}_0^{\infty}$ denote the sequence of its partial sums. Let $\{q_n\}_0^{\infty}$ be a sequence of real numbers with $q_0 > 0$, and $q_n \geq 0$ for all $n > 0$, and let $Q_n = \sum_{k=0}^n q_k$. By G. H. Hardy [6] the sequence-to-sequence transformation

$$T_n = \frac{1}{Q_n} \sum_{k=0}^n q_{n-k} S_k$$

is called the *Norlund means* of $\{S_n\}_0^{\infty}$, and is denoted by (N, q_n) .

The (N, q_n) is regular if and only if $q_n = o(Q_n)$ as $n \rightarrow \infty$; furthermore, the sequence-to-sequence transformation

$$\overline{T}_n = \frac{1}{Q_n} \sum_{k=0}^n q_k S_k$$

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is called the *progressive means* of $\{S_n\}_0^\infty$, and is denoted by $(\overline{N}, q_n)$. The $(\overline{N}, q_n)$ is regular if and only if $Q_n \rightarrow \infty$ as $n \rightarrow \infty$. By Peter L. Duren [4] a function f analytic in a domain D is said to be simple, schlicht, or univalent if f is one-to-one mapping of D onto another domain. A domain E of the complex plane is said to be convex if and only if the line segment joining any two points of E lies entirely in E . A function f which is analytic, univalent in the unit disc $D = \{z : |z| < 1\}$, and is normalized by $f(0) = f'(0) - 1 = 0$ is said to belong to the class S . Now $f \in S$ is said to belong to the class K if and only if it is a conformal mapping of the unit disc $D = \{z : |z| < 1\}$ onto a convex domain. An analytic function g is said to be subordinate to an analytic function f (written $g \prec f$) if

$$g(z) = f(\omega(z)) \quad |z| < 1$$

for some analytic function ω with $|\omega(z)| \leq |z|$. It is known by the Koebe-One-Quarter theorem that the range of every function of the class S contains the disc $\{w : |w| < \frac{1}{4}\}$, i.e. $\frac{1}{4}z \prec f$. The strengthened version of the Koebe-One-Quarter theorem says that the range of every convex function $f \in K$ contains the disc $|w| < \frac{1}{2}$, i.e. $\frac{1}{2}z \prec f$. The Chebychev's polynomials of the first kind $T_n(x)$, and of the second kind $U_n(x)$ are respectively defined by:

$$T_n(x) = \cos n\theta, \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad x = \cos \theta.$$

2. MEANS CONNECTED WITH POWER SERIES

Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$. Let

- $S_n(z, f) = \sum_{k=0}^n a_k z^k$ be the sequence of partial sums of f ,
- $\sigma_n(z, f) = \frac{1}{n+1} \sum_{k=0}^n S_k(z, f)$ be the Cesaro means or $(C, 1)$ means of f ,
- $T_n(z, f) = \frac{1}{Q_n} \sum_{k=0}^n q_{n-k} S_k(z, f)$ be the Norlund means of f ,
- $\overline{T}_n(z, f) = \frac{1}{Q_n} \sum_{k=0}^n q_k S_k(z, f)$ be the Progressive means of f ,
- $V_n(z, f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^n \binom{2n}{n+k} a_k z^k$ be the de la Vallee Poussin means of f .

3. KNOWN RESULTS

In [7] G. Pólya and I. J. Schoenberg proved the following theorem, and corollary:

Theorem 3.1. *For $f(z) \in K$, it is necessary and sufficient that $V_n(z, f) \in K$ for $n = 1, 2, \dots$.*

Corollary 3.2. *For $f(z) \in K$, $V_n(z, f) \prec f$ for $n = 1, 2, \dots$.*

In [3] T. Başgöze, J. L. Frank, and F. R. Keogh proved the following theorem:

Theorem 3.3. (i) Suppose that the values taken by $f(z)$ for z in D lie in a convex domain D_w . Then the values taken by $\sigma_n(z, f)$ also lie in D_w for all n , and all z in D .
(ii) Conversely, suppose that the values taken by $\sigma_n(z, f)$ lie in a convex domain D_w ; then the values taken by $f(z)$ lie in D_w for all z in D .

In [1] Ziad S. Ali proved the following theorems:

Theorem 3.4. (i) Let (N, q_n) be a regular Norlund transformation such that $\{q_n\}_0^\infty$ is a non-decreasing sequence of positive numbers. Suppose that the values taken by $f(z)$, for z in D , lie in a convex domain D_w , then the values taken by $T_n(z, f)$, also lie in D_w for all n , and all z in D .
(ii) Conversely, suppose that the values taken by $T_n(z, f)$ lie in a convex domain D_w ; then the values taken by $f(z)$ lie in D_w for all z in D .

Theorem 3.5. (i) Let $(\overline{N}, q_n)$ be a regular Progressive transformation such that $\{q_n\}_0^\infty$ is a non-increasing sequence of positive numbers. Suppose that the values taken by $f(z)$, for z in D , lie in a convex domain D_w , then the values taken by $\overline{T}_n(z, f)$, also lie in D_w for all n , and all z in D .
(ii) Conversely, suppose that the values taken by $\overline{T}_n(z, f)$ lie in a convex domain D_w ; then the values taken by $f(z)$ lie in D_w for all z in D .

In [2] Ziad S. Ali proved the following theorem:

Theorem 3.6. (i) Let $f(z) = \sum_{k=1}^\infty a_k z^k$, ($c_1 = 1$) be regular in the unit disc $|z| < 1$.
(ii) Let T_n be a transformation of the Norlund type. Let

$$Q_k^n = \sum_{r=0}^k q_r^n = \sum_{r=0}^k \frac{(2n - 2r + 1)}{(2n - r + 1)} \binom{2n}{r} q_0,$$

and

$$\omega(n) = \frac{-2}{Q_n^n} \sum_{k=1}^n (-1)^k Q_{n-k}^n,$$

then $\frac{1}{\omega(n)} T_n(z, f) \in K$ if and only if $f \in K$.

4. THE MAIN THEOREMS

In this section we prove the following theorems:

Theorem 4.1. (i) Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$, ($a_1 = 1$) be regular in the unit disc $|z| < 1$, and let $\overline{T}_n(z, f) = \frac{1}{Q_n} \sum_{k=0}^n q_k S_k(z, f)$ be a transformation of the progressive type.

(ii) Let

$$Q_{n-k} = Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n, \text{ and } Q_n = Q_n^n = \sum_{r=0}^n q_r^n,$$

$$q_r^n = \begin{cases} \frac{(2n-2r+1)}{(2n-r+1)} \binom{2n}{r} q_0 & \text{if } r = 0, 1, \dots, (n-k), \\ q_{n-r}^n & \text{if } r = (n-k) + 1, (n-k) + 2, \dots, n-1, n. \end{cases}$$

(iii) Let

$$\overline{\omega}(n) = \frac{-2}{Q_n^n} \sum_{k=1}^n (-1)^k (Q_n^n - Q_{k-1}^n),$$

then $\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) \in K$ if and only if $f \in K$.

Proof. We begin first by noting that:

$$\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) = \frac{1}{\frac{-2}{Q_n^n} \sum_{k=1}^n (-1)^k (Q_n^n - Q_{k-1}^n)} \frac{1}{Q_n^n} \sum_{k=1}^n q_k^n S_k(z, f)$$

expanding $\sum_{k=1}^n q_k^n S_k(z, f)$, we can easily see:

$$\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) = \frac{1}{-2 \sum_{k=1}^n (-1)^k (Q_n^n - Q_{k-1}^n)} \sum_{k=1}^n (Q_n^n - Q_{k-1}^n) a_k z^k.$$

Since

$$Q_n^n = Q_{(n-k)}^n + (q_{(n-k)+1}^n + q_{(n-k)+2}^n + \dots + q_{n-1}^n + q_n^n), \quad \text{and} \\ Q_{k-1}^n = q_{k-1}^n + q_{k-2}^n + \dots + q_1^n + q_0^n.$$

Hence

$$\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) = \frac{\sum_{k=1}^n \left((Q_{(n-k)}^n + (q_{(n-k)+1}^n + \dots + q_{n-1}^n + q_n^n)) - (q_{k-1}^n + q_{k-2}^n + \dots + q_1^n + q_0^n) \right) a_k z^k}{-2 \sum_{k=1}^n (-1)^k \left((Q_{(n-k)}^n + (q_{(n-k)+1}^n + \dots + q_{n-1}^n + q_n^n)) - (q_{k-1}^n + q_{k-2}^n + \dots + q_1^n + q_0^n) \right)}.$$

Equivalently we have:

$$\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) =$$

$$\frac{\sum_{k=1}^n (Q_{(n-k)}^n + (q_{n-(k-1)}^n + \cdots + q_{n-1}^n + q_n^n) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n)) a_k z^k}{-2 \sum_{k=1}^n (-1)^k (Q_{(n-k)}^n + (q_{n-(k-1)}^n + \cdots + q_{n-1}^n + q_n^n) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n))}.$$

Since $q_r^n = q_{n-r}^n$ for $r = n - (k - 1), n - (k - 2), \dots, (n - 1), n$, it follows easily that:

$$(q_{n-(k-1)}^n + q_{n-(k-2)}^n + \cdots + q_{n-1}^n + q_n^n) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n) = 0.$$

Hence

$$\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) = \frac{1}{-2 \sum_{k=1}^n (-1)^k Q_{(n-k)}^n} \sum_{k=1}^n Q_{(n-k)}^n a_k z^k.$$

Now we can easily show that:

$$Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n = \sum_{r=0}^{n-k} \frac{(2n - 2r + 1)}{(2n - r + 1)} \binom{2n}{r} q_0 = \binom{2n}{n-k} q_0.$$

Hence

$$\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) = \frac{1}{-2 \sum_{k=1}^n (-1)^k \binom{2n}{n-k}} \sum_{k=1}^n \binom{2n}{n-k} a_k z^k.$$

Finally we can show that for n odd we have:

$$-2 \sum_{k=1}^n (-1)^k \binom{2n}{n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} + \binom{2n}{n},$$

and for n even we have:

$$-2 \sum_{k=1}^n (-1)^k \binom{2n}{n-k} = - \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} + \binom{2n}{n}.$$

Now since:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} = 0.$$

It follows that

$$\frac{1}{\overline{\omega}(n)} \overline{T}_n(z, f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^n \binom{2n}{n+k} a_k z^k = V_n(z, f),$$

which are the de la Vallee Poussin means of f , and the theorem follows by G. Pólya and I. J. Schoenberg [7]. $\square$

Theorem 4.2. (i) Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$, and suppose that $\overline{T}_n$ are the Progressive means.

(ii) Let $Q_n^n = n + 1$, and let

$$\bar{\omega}(n) = \begin{cases} \frac{-2}{Q_n^n} \sum_{k=1}^n (-1)^k (Q_n^n - Q_{k-1}^n) & n \text{ is odd} \\ \frac{-2}{Q_n^n} \sum_{k=1}^n (-1)^k (Q_n^n - Q_{k-1}^n) + 1 & n \text{ is even,} \end{cases}$$

then

$$\frac{1}{\bar{\omega}(n)} \bar{T}_n(z, f) \in K \quad \text{if and only if } f \in K.$$

Proof. Clearly $Q_n^n - Q_{k-1}^n = n - k + 1$. Considering two separate cases for n even, and n odd we can easily see that

$$n + 1 = \begin{cases} -2 \sum_{k=1}^n (-1)^k (n - k + 1) & n \text{ is odd} \\ -2 \sum_{k=1}^n (-1)^k (n - k + 1) + 1 & n \text{ is even.} \end{cases}$$

Accordingly for any n we have:

$$\frac{1}{\bar{\omega}(n)} \bar{T}_n(z, f) = \frac{1}{n + 1} \sum_{k=0}^n S_k(z, f) = \sigma_n(z, f),$$

which are the Cesaro means of f , and the result follows by T. Başgöze, J. L. Frank, and F. R. Keogh [3]. $\square$

Theorem 4.3. (i) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be regular in the unit disc $D = \{z : |z| < 1\}$.

(ii) Let $\bar{T}_n(z, f)$ be a regular Progressive type transformation defined by a non-increasing sequence $\{q_r^n\}_{r=1}$ of positive real numbers such that $\sum_{i \in \text{odd}} q_i^n = \sum_{i \in \text{even}} q_i^n$, where i is a non-negative integer then:

$$\frac{1}{\bar{\omega}(n)} \bar{T}_n(z, f) \in K \quad \text{if and only if } f \in K.$$

Proof. For n odd integer, say $n = 2s + 1$ we have:

$$\begin{aligned} -2 \sum_{i=1}^n (-1)^i (Q_n^n - Q_{i-1}^n) &= -2 \sum_{i=1}^{2s+1} (-1)^i (Q_{2s+1}^{2s+1} - Q_{i-1}^{2s+1}) \\ &= -2 \left(- \sum_{t=0}^s q_{2t+1}^{2s+1} \right) = 2 \sum_{i \in \text{odd}}^n q_i^n, \quad i = 1, 3, 5, \dots \end{aligned}$$

Similarly for $n = 2s$ we have:

$$\begin{aligned} -2 \sum_{i=1}^n (-1)^i (Q_n^n - Q_{i-1}^n) &= -2 \sum_{i=1}^{2s} (-1)^i (Q_{2s}^{2s} - Q_{i-1}^{2s}) \\ &= -2 \left(- \sum_{t=0}^{s-1} q_{2t+1}^{2s} \right) = 2 \sum_{i \in \text{odd}}^n q_i^n, \quad i = 1, 3, 5, \dots \end{aligned}$$

Therefore,

$$\bar{\omega}(n) = \frac{-2}{Q_n^n} \sum_{i=1}^n (-1)^i (Q_n^n - Q_{i-1}^n) = \frac{1}{Q_n^n} \left(\sum_{i \in \text{odd}}^n q_i^n + \sum_{i \in \text{even}}^n q_i^n \right) = 1.$$

Accordingly the result follows by Ziad S. Ali [1]. $\square$

5. THEOREMS ON $\bar{\omega}(n)$

In this section we see more of the properties of $\bar{\omega}(n)$ through the following theorems.

Theorem 5.1. *Let $\{q_r^n\}_{r=1}^n$ be a sequence of positive real numbers, then*

$$\bar{\omega}(n) = 1 \quad \text{if and only if} \quad \sum_{r \in \text{odd}} q_r^n = \sum_{r \in \text{even}} q_r^n.$$

Proof. Let $\bar{\omega}(n) = 1$, then

$$-2 \sum_{r=1}^n (-1)_r (Q_n^n - Q_r^n) = Q_n^n \quad 2 \sum_{r \in \text{odd}} q_r^n = \left(\sum_{r \in \text{odd}}^n q_r^n + \sum_{r \in \text{even}}^n q_r^n \right).$$

Now assume $\sum_{r \in \text{odd}}^n q_r^n = \sum_{r \in \text{even}}^n q_r^n$, then

$$\bar{\omega}(n) = \frac{1}{Q_n^n} \left(\sum_{r \in \text{even}}^n q_r^n + \sum_{r \in \text{odd}}^n q_r^n \right) = 1. \quad \square$$

Theorem 5.1 can be used as a tool to generate or prove new Combinatorial identities as seen by the following theorems:

Theorem 5.2. *Let $q_r^n = \binom{n}{r}$, then*

$$\sum_{r \in \text{odd}} q_r^n = \sum_{r \in \text{even}} q_r^n, \quad \text{and} \quad -\frac{1}{2^{n-1}} \sum_{r=1}^n (-1)^r \left(2^n - \sum_{j=0}^{r-1} \binom{n}{j} \right) = 1.$$

Proof. With $q_r^n = \binom{n}{r}$, we have:

$$\bar{\omega}(n) = -\frac{2}{Q_n^n} \left(\sum_{r=1}^n (-1)^r (Q_n^n - Q_{r-1}^n) \right) = \frac{1}{Q_n^n} \left(\sum_{r \in \text{odd}} \binom{n}{r} + \sum_{r \in \text{even}} \binom{n}{r} \right) = 1.$$

Accordingly by theorem 5.1 we have the following combinatorial identity:

$$-\frac{1}{2^{n-1}} \sum_{r=1}^n (-1)^r \left(2^n - \sum_{j=0}^{r-1} \binom{n}{j} \right) = 1.$$

The above newly generated combinatorial identity is implicitly saying for example when n is even: The sum of all combinations of n elements taken r at a time with $r = 1, 3, 5, \dots$ is 2^{n-1} . $\square$

Theorem 5.3. Let $q_r^n = \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} q_0$, and $Q_n^n = \sum_{r=0}^n q_r^n$. Then we have:

$$\sum_{r \in \text{odd}}^n q_r^n = \sum_{r \in \text{even}}^n q_r^n.$$

Proof. With $q_r^n = \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} q_0$, we can show:

$$\begin{aligned} \bar{\omega}(n) &= \frac{-2}{Q_n^n} \left(\sum_{r=1}^n (-1)^r (Q_n^n - Q_{r-1}^n) \right) = \frac{2}{Q_n^n} \sum_{r \in \text{odd}}^n \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} \\ &= \frac{2}{Q_n^n} \sum_{r \in \text{even}}^n \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} = 1. \end{aligned}$$

Now we may apply theorem 5.1. Moreover for n even we have:

$$\begin{aligned} \sum_{r \in \text{even}}^n \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} &= \sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} - \sum_{r=0}^{\frac{n}{2}-1} \binom{2n}{2r+1} = \frac{1}{2} \binom{2n}{n} \\ \sum_{r \in \text{odd}}^n \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} &= \sum_{r=0}^{\frac{n}{2}-1} \binom{2n}{2r+1} - \sum_{r=0}^{\frac{n}{2}-1} \binom{2n}{2r} = \frac{1}{2} \binom{2n}{n}. \end{aligned}$$

For $n \in \text{odd}$ we have:

$$\begin{aligned} \sum_{r \in \text{even}} \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} &= \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r} - \sum_{r=0}^{\frac{n-3}{2}} \binom{2n}{2r+1} = \frac{1}{2} \binom{2n}{n} \\ \sum_{r \in \text{odd}} \frac{2n-2r+1}{2n-r+1} \binom{2n}{r} &= \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} - \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r} = \frac{1}{2} \binom{2n}{n}. \end{aligned}$$

This completes the proof of theorem 5.3. $\square$

We note from theorem 5.3. above that for n even

$$\sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = \sum_{r=0}^{\frac{n}{2}-1} \binom{2n}{2r+1} + \binom{2n-1}{n} \quad ; \quad \sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = 2^{2n-2} + \binom{2n-1}{n}.$$

For n odd we can show

$$\sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = 2^{2n-2}.$$

Accordingly for any n we have:

$$\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{2r} = 2^{2n-2} + \frac{1 + (-1)^n}{2} \binom{2n-1}{n}, \quad n \geq 1,$$

which is identity 1.92 of Henry W. Gould [5]. Similarly for n even we have from theorem 5.3:

$$\sum_{r=0}^{\frac{n}{2}-1} \binom{2n}{2r+1} = 2^{2n-2}.$$

Now for n odd we have:

$$\sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} = \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r} + \binom{2n-1}{n} = 2^{2n-2} + \binom{2n-1}{n}.$$

Accordingly for any n we have:

$$\sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{2n}{2r+1} = 2^{2n-2} + \frac{1 - (-1)^n}{2} \binom{2n-1}{n},$$

which is identity 1.98 of Henry W. Gould [5].

Theorem 5.4. *For $n > 1$ we have:*

$$-2 \left(\sum_{k=1}^n (-1)^k \cdot \left(\sum_{r=k}^n r^2 \binom{2n}{n-r} \right) \right) = \sum_{r=1}^n r^2 \binom{2n}{n-r}.$$

Proof. Follows by theorem 5.1 and noting that for $n > 1$ we have:

$$\sum_{\substack{r \in \text{odd} \\ r \geq 1}}^n r^2 \binom{2n}{n-r} = \sum_{\substack{r \in \text{even} \\ r \geq 2}}^n r^2 \binom{2n}{n-r}.$$

□

6. A KEY LEMMA

In this section we have the following lemma:

Lemma 6.1. *For $1 \leq r \leq n$, and θ real we have:*

- (i) $\sum_{r=1}^n \binom{2n}{n-r} (\cos r\theta + (-1)^{r+1}) = 2^{n-1} (1 + \cos \theta)^n.$
- (ii) $\sum_{r=1}^n (-1)^r \binom{2n}{n-r} \cos r\theta + \frac{1}{2} \binom{2n}{n} = 2^{n-1} (1 - \cos \theta)^n.$

Proof. (i) Using induction on n , and by repeated application of the recurrence formula

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1},$$

the above lemma follows. Note now that the proof of the lemma also follows by noting that:

$$\cos^{2n} \frac{\theta}{2} = \frac{1}{2^n} \binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{r=0}^{n-1} \binom{2n}{r} \cos(n-r)\theta,$$

and where

$$\sum_{r=1}^n (-1)^{r+1} \binom{2n}{n-r} = \frac{1}{2} \binom{2n}{n}.$$

Furthermore note that the above lemma also follows from the following:

$$\operatorname{Re} \left(e^{in\theta} (1 + e^{-i\theta})^{2n} \right) = \sum_{r=0}^{2n} \binom{2n}{r} \cos(n-r)\theta = 2^{2n} \cos^{2n} \frac{\theta}{2}.$$

We can also see that

$$\sum_{r=0}^n \binom{2n}{r} \cos(n-r)\theta = \frac{1}{2} \sum_{r=0}^{2n} \binom{2n}{r} \cos(n-r)\theta + \frac{1}{2} \binom{2n}{n}.$$

Now with $k = n - r$ it follows that

$$\sum_{k=1}^n \binom{2n}{n-k} \cos k\theta + \frac{1}{2} \binom{2n}{n} = 2^{2n-1} \cos^{2n} \frac{\theta}{2}.$$

Accordingly

$$\sum_{r=1}^n \binom{2n}{n-r} (\cos r\theta + (-1)^{r+1}) = 2^{n-1} (\cos \theta + 1)^n,$$

and the lemma follows again.

(ii) Follows since

$$\begin{aligned} (e^{-in\theta} (1 - e^{i\theta})^{2n}) &= 2^{2n} \sin^{2n} \frac{\theta}{2} \cdot (-1)^n \\ &= (-1)^n \left(\binom{2n}{n} + 2 \sum_{r=1}^n (-1)^r \binom{2n}{n-r} \cos r\theta \right), \\ &= \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} \cos(n-r)\theta. \end{aligned}$$

This completes the proof of lemma 6.1. □

Remark 1. From above we have for $m = 2n$

$$\sum_{r=0}^m \binom{m}{r} \cos\left(\frac{m}{2} - r\right)\theta = 2^m \cos^m \frac{\theta}{2} \cdot 1.$$

Accordingly we have:

$$\sum_{r=0}^m \binom{m}{r} \cos r\theta = 2^m \cos \frac{m\theta}{2} \cos^m \frac{\theta}{2}$$

$$\sum_{r=0}^m \binom{m}{r} \sin r\theta = 2^m \sin \frac{m\theta}{2} \cos^m \frac{\theta}{2}.$$

For any m the above two combinatorial identities which are 1.26, and 1.27 in the list of identities of Henry W. Gould [5] follow by considering $(1 + e^{i\theta})^m$.

Remark 2. Similarly for $m = 2n$ we have:

$$\sum_{r=0}^m (-1)^r \binom{m}{r} \cos\left(\frac{m}{2} - r\right)\theta = (-1)^{\frac{m}{2}} 2^m \sin^m \frac{\theta}{2} \cdot 1$$

then we have:

$$\sum_{r=0}^m (-1)^r \binom{m}{r} \cos r\theta = (-1)^{\frac{m}{2}} 2^m \sin^m \frac{\theta}{2} \cos \frac{m\theta}{2}$$

$$\sum_{r=0}^m (-1)^r \binom{m}{r} \sin r\theta = (-1)^{\frac{m}{2}} 2^m \sin^m \frac{\theta}{2} \sin \frac{m\theta}{2}.$$

For any m the above two combinatorial identities which are 1.28, and 1.29 of the identities of Henry W. Gould [5] follow by considering $(1 - e^{i\theta})^m$. Now for $\theta = 0$ in lemma 6.1(i) we can show the following:

$$\sum_{r=0}^n \binom{2n}{r} = 2^{2n-1} + \binom{2n-1}{n}$$

$$\sum_{r=0}^n (-1)^r \binom{2n}{r} = (-1)^n \binom{2n-1}{n}$$

$$\sum_{r=0}^n \binom{2n+1}{r} = 4^n$$

which are 1.85, 1.86, and 1.83 of Henry W. Gould [5].

Corollary 6.2. *For $\theta \in \text{real}$, and $r \leq n$ we have the following combinatorial trigonometric identities:*

$$\sum_{r \in \text{even}}^n \binom{2n}{n-r} \cos r\theta = 2^{2n-2} \left(\cos^{2n} \frac{\theta}{2} + \sin^{2n} \frac{\theta}{2} \right) - \frac{1}{2} \binom{2n}{n}$$

$$\sum_{r \in \text{odd}}^n \binom{2n}{n-r} \cos r\theta = 2^{2n-2} \left(\cos^{2n} \frac{\theta}{2} - \sin^{2n} \frac{\theta}{2} \right).$$

Proof. Follows from lemma 6.1. □

Corollary 6.3. *For $r \leq n$ we have:*

$$\sum_{\substack{r \in \text{odd} \\ r \geq 1}}^n \binom{2n}{n-r} r \cdot \sin r\theta = n2^{n-1} \sin \theta \cdot \left(\sum_{\substack{r \in \text{even} \\ r \geq 0}}^{n-1} \binom{n-1}{r} \cos^r \theta \right), \quad n \geq 1$$

$$\sum_{\substack{r \in \text{even} \\ r > 1}}^n \binom{2n}{n-r} r \cdot \sin r\theta = n2^{n-1} \sin \theta \cdot \left(\sum_{\substack{r \in \text{odd} \\ r \geq 1}}^{n-1} \binom{n-1}{r} \cos^r \theta \right), \quad n \geq 1.$$

Proof. Follows from lemma 6.1. □

Corollary 6.4. *For $n \geq 2$ we have:*

$$\sum_{\substack{r \in \text{odd} \\ r \geq 1}}^n \binom{2n}{n-r} r^2 \cdot \cos r\theta$$

$$= n \cdot 2^{n-1} \left((1 - n \sin^2 \theta) \cdot \sum_{\substack{r \in \text{odd} \\ r \geq 1}}^{n-2} \binom{n-2}{r} \cdot \cos^r \theta + \sum_{\substack{r \in \text{even} \\ r \geq 0}}^{n-2} \binom{n-2}{r} \cos^{r+1} \theta \right)$$

$$\sum_{\substack{r \in \text{even} \\ r \geq 2}}^n \binom{2n}{n-r} r^2 \cdot \cos r\theta$$

$$= n \cdot 2^{n-1} \left((1 - n \sin^2 \theta) \cdot \sum_{\substack{r \in \text{even} \\ r \geq 0}}^{n-2} \binom{n-2}{r} \cdot \cos^r \theta + \sum_{\substack{r \in \text{odd} \\ r \geq 1}}^{n-2} \binom{n-2}{r} \cos^{r+1} \theta \right).$$

Proof. Follows from lemma 6.1. □

Corollary 6.5. *For $0 \leq r \leq n$ we have:*

$$\sum_{r=0}^n r \binom{2n}{r} = n \cdot 2^{2n-1}$$

$$\sum_{r=0}^n r^2 \binom{2n}{r} = n \cdot 2^{2n-2} + n^2 2^{2n-1} - n^2 \binom{2n-1}{n}.$$

Proof. Since from lemma 6.1 we have:

$$\sum_{r=1}^n r^2 \binom{2n}{n-r} = n \cdot 2^{2n-2}.$$

Furthermore since we can also show that

$$\sum_{r=0}^n r \binom{2n}{n+r} = \frac{n}{2} \binom{2n}{n},$$

the corollary follows. □

7. GENERATING THE CHEBYSHEV'S POLYNOMIALS

Using lemma 6.1(i), then by the definition of the Chebyshev's polynomials of the first kind $T_n(x)$, we see that $T_n(x)$ satisfies the following formula:

$$\sum_{r=1}^n \binom{2n}{n-r} (T_r(x) + (-1)^{r+1}) = 2^{n-1}(x+1)^n, \quad x = \cos \theta.$$

Now by letting $r = 1, r = 2, r = 3, \dots$ etc. we can respectively obtain

$$T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \dots$$

hence generating the Chebyshev's polynomials of the first kind of degrees $1, 2, 3, \dots$ etc. We can similarly see that $U_n(x)$, the Chebyshev's polynomials of the second kind satisfy:

$$\sum_{r=1}^n \binom{2n}{n-r} \cdot r \cdot U_{r-1}(x) = n \cdot 2^{n-1}(x+1)^{n-1}, \quad x = \cos \theta.$$

Again now for $r = 1, r = 2, r = 3, \dots$ etc. we can respectively obtain

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \dots,$$

and hence generating the Chebyshev's polynomials of the second kind of degrees $0, 1, 2, \dots$ etc.

8. AN APPLICATION ON PROBABILITIES

Using lemma 6.1(i), we can show that the probability of n successes in $2n$ trials of a symmetric binomial distribution is given by:

$$(1) \quad \frac{\binom{2n}{n}}{2^{2n}} = \frac{1}{2^{2n-1}} \sum_{r=0}^n \binom{2n}{r} \cos \frac{(n-r)\pi}{2} - \frac{1}{2^n}$$

$$(2) \quad \frac{\binom{2n}{n}}{2^{2n}} = \frac{2 \sum_{r=0}^n \binom{2n}{r}}{2^{2n}} - 1$$

$$(3) \quad \frac{\binom{2n}{n}}{2^{2n}} = \frac{1}{2\pi} \int_0^{2\pi} \cos^{2n} t \, dt$$

$$(4) \quad \frac{\binom{2n}{n}}{2^{2n}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}.$$

9. A DIFFERENT FORM OF $\overline{\omega}(n)$

A different form of $\overline{\omega}(n)$ is presented in this section, and this is seen by the following:

Theorem 9.1. (i) Let $f(z) = \sum_{k=1}^{\infty} c_k z^k$ ($c_1 = 1$) be regular in the unit disc

$$|z| < 1.$$

(ii) Let

$$Q_{n-k} = Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n, \text{ and } Q_n = Q_n^n = \sum_{r=0}^n q_r^n,$$

$$q_r^n = \begin{cases} \frac{(2n-2r+1)}{(2n-r+1)} \binom{2n}{r} q_0 & r = 0, 1, \dots, (n-k), \\ q_{n-r}^n & r = (n-k) + 1, (n-k) + 2, \dots, n-1, n. \end{cases}$$

(iii) Let $\overline{T}_n$ be the Progressive means. With $z = \rho e^{i\theta}$, let

$$\overline{\omega}_m(n, \theta) = \frac{-2}{Q_n^n} \min_{|z| \leq 1} \operatorname{Re} \cdot \sum_{r=1}^n (Q_n^n - Q_{r-1}^n) \cdot z^r, \text{ then}$$

$$\frac{1}{\overline{\omega}_m(n, \theta)} \overline{T}_n(z, f) \in K \text{ if and only if } f \in K.$$

Proof. $u(\rho, \theta) = \sum_{k=1}^n \binom{2n}{n-k} \rho^k \cos k\theta$ is harmonic in

$$D = \{z : |z| < 1\} \quad \text{as} \quad \nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} = 0.$$

Furthermore u is continuous on $\overline{D} : \{z : |z| \leq 1\}$. Accordingly by the minimum principle for harmonic functions u attains its minimum on the boundary of D . Now the proof of theorem 9.1 follows from lemma 6.1(i), and theorem 4.1.

Note that from lemma 6.1(i), or $-\sum_{k=1}^n \binom{2n}{n-k} k \sin k\theta$ guarantees a minimum at $\theta = \pi \in [0, 2\pi]$. $\square$

10. THE SUBORDINATION PRINCIPLE AND $\bar{\omega}_m(n, \theta)$

In this section we relate $\bar{\omega}(n)$ to the subordination principle by the following theorem.

Theorem 10.1. (i) Let K denote the class of “Schlicht” power series which map $|z| < 1$ onto some convex domain, and let $f \in K$.

(ii) Let

$$Q_{n-k} = Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n, \text{ and } Q_n = Q_n^n = \sum_{r=0}^n q_r^n,$$

$$q_r^n = \begin{cases} \frac{(2n-2r+1)}{(2n-r+1)} \binom{2n}{r} q_0 & r = 0, 1, \dots, (n-k), \\ q_{n-r}^n & r = (n-k) + 1, (n-k) + 2, \dots, n-1, n. \end{cases}$$

(iii) Let $\bar{T}_n$ be a transformation of the Progressive type. With $z = \rho e^{i\theta}$, let

$$\bar{\omega}_m(n, \theta) = \frac{-2}{Q_n^n} \min_{|z| \leq 1} \operatorname{Re} \sum_{r=1}^n (Q_n^n - Q_{r-1}^n) \cdot z^r, \text{ then}$$

$$\frac{1}{\bar{\omega}_m(n, \theta)} \bar{T}_n(z, f) \prec f.$$

Proof. Follows from the proof of 9.1, and corollary 3.2 of G. Pólya and I. J. Schoenberg [7]. Note that

$$\frac{1}{\bar{\omega}_m(1, \theta)} \bar{T}_1(z, f) = \frac{1}{2} z \prec f,$$

which is the strengthened version of the Koebe-One-Quarter theorem, and

$$\frac{1}{\bar{\omega}_m(2, \theta)} \bar{T}_2(z, f) = \frac{2}{3} z + \frac{a_2}{6} z^2 = V_2(z, f) \prec f. \quad \square$$

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