

SEMI-INVARIANT PRODUCTS OF A NEARLY SASAKIAN MANIFOLD

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ABSTRACT. In this paper, we give some sufficient conditions for semi-invariant product of a nearly Sasakian manifold, and generalize Bejancu's result.

1. INTRODUCTION

Let $\overline{M}$ be a real $(2n + 1)$ -dimensional almost contact metric manifold with the structure tensors (Φ, ξ, η, g) , then

$$(1.1) \quad \Phi\xi = 0, \eta(\xi) = 1, \Phi^2 = -I + \eta \otimes \xi, \eta(X) = g(X, \xi),$$

$$(1.2) \quad g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta \circ \Phi = 0,$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.1 ([1]). The Nijenhuis tensor field of Φ on an almost contact metric manifold is defined by

$$(1.3) \quad [\Phi, \Phi](X, Y) = [\Phi X, \Phi Y] + \Phi^2[X, Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y],$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.2 ([1]). An almost contact metric manifold $\overline{M}$ is called a nearly Sasakian manifold, if we have

$$(1.4) \quad (\overline{\nabla}_X \Phi)Y + (\overline{\nabla}_Y \Phi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y,$$

for any $X, Y \in \Gamma(T\overline{M})$.

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Definition 1.3 ([1]). An almost contact metric manifold $\overline{M}$ is called a Sasakian manifold, if we have

$$(1.5) \quad (\overline{\nabla}_X \Phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any $X, Y \in \Gamma(T\overline{M})$.

Obviously, a Sasakian manifold is a nearly Sasakian manifold.

Let M be an m -dimensional submanifold of an n -dimensional almost contact metric manifold $\overline{M}$. We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, denote by ∇ the induced connection on M , and denote by $\nabla^\perp$ the normal connection on M . Thus, for any $X, Y \in \Gamma(TM)$, we have

$$(1.6) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where $h : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM^\perp)$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.6) is called the Gauss formula and h is called the second fundamental form of M .

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, we denote by $-A_V X$ and $\nabla_X^\perp V$ the tangent part and normal part of $\overline{\nabla}_X V$ respectively. Then we have

$$(1.7) \quad \overline{\nabla}_X V = -A_V X + \nabla_X^\perp V.$$

Thus, for any $V \in \Gamma(TM^\perp)$, we have a linear operator, satisfying

$$(1.8) \quad g(A_V X, Y) = g(X, A_V Y) = g(h(X, Y), V).$$

The equation (1.7) is called the Weingarten formula.

An m -dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assigns to each point x of $\overline{M}$ an m -dimensional linear subspace D_x of $T_x \overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x \in D_x$ for each $x \in \overline{M}$. When this happens we write $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $x \in \overline{M}$ there exist m differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of x . From now on, all distribution are supposed to be differentiable of class C^∞ .

The distribution D is said to be involutive if for all vector fields $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$. A submanifold M of $\overline{M}$ is said to be an integral manifold of D if for every point $x \in M$, D_x coincides with the tangent space to M at x . If there exists no integral manifold of D which contains M , then M is called a maximal integral manifold or a leaf of D . The distribution D is said to be integrable if for every $x \in \overline{M}$ there exists an integral manifold of D containing x .

Definition 1.4 ([1]). Let M be a real $(2m+1)$ -dimensional submanifold of a real $(2n+1)$ -dimensional almost contact metric manifold $\overline{M}$ with the structure tensors (Φ, ξ, η, g) . We assume that the structure tensor ξ is tangent to M , and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M . Then M is called a semi-invariant submanifold of $\overline{M}$, if there exist two differentiable distributions D and $D^\perp$ on M , satisfying

- (1) $TM = D \oplus D^\perp \oplus \{\xi\}$, where D , $D^\perp$ and $\{\xi\}$ are mutually orthogonal to each other;
- (2) the distribution D is invariant by Φ , that is, $\Phi(D_x) = D_x$, for each $x \in M$;
- (3) the distribution $D^\perp$ is anti-invariant by Φ , that is, $\Phi(D_x^\perp) \subset T_x M^\perp$, for each $x \in M$.

Let M be a semi-invariant submanifold of a nearly Sasakian Manifold $\overline{M}$. Then the normal bundle to M has the orthogonal decomposition

$$(1.9) \quad TM^\perp = JD^\perp \oplus \nu.$$

For each vector field X tangent to M , we put

$$(1.10) \quad \Phi X = \psi X + \omega X,$$

where ψX and ωX are respectively the tangent part and the normal part of ΦX . Also, for each vector field V normal to M , we put

$$(1.11) \quad \Phi V = BV + CV,$$

where BV and CV are respectively the tangent part and the normal part of ΦV .

We denote by P and Q the projection morphisms of TM on D and $D^\perp$ respectively. Then

$$(1.12) \quad X = PX + QX + \eta(X)\xi,$$

for any $X \in \Gamma(TM)$.

Definition 1.5 ([1]). Let M be a semi-invariant submanifold of a nearly Sasakian Manifold $\overline{M}$. We say that M is a semi-invariant product if the distribution $D \oplus \{\xi\}$ and the distribution $D^\perp$ are integrable and locally M is a Riemannian product $M_1 \times M_2$, where M_1 (resp. M_2) is a leaf of $D \oplus \{\xi\}$ (resp. $D^\perp$)

In the references [1], we know that a distribution D on M is integrable if and only if $[X, Y] \in \Gamma(D)$ for all vector fields $X, Y \in \Gamma(D)$.

2. MAIN RESULTS

Lemma 2.1. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then*

$$(2.1) \quad \begin{aligned} \nabla_X \psi Y - A_{\omega Y} X + \nabla_Y \psi X - A_{\omega X} Y = \\ = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y + \psi \nabla_X Y + \psi \nabla_Y X + 2Bh(X, Y), \end{aligned}$$

$$(2.2) \quad \eta(\nabla_X \psi Y) + \eta(\nabla_Y \psi X) = 2g(\Phi X, \Phi Y) + \eta(A_{\omega Y} X) + \eta(A_{\omega X} Y),$$

$$(2.3) \quad h(X, \psi Y) + \nabla_X^\perp \omega Y + h(Y, \psi X) + \nabla_Y^\perp \omega X = \omega \nabla_X Y + \omega \nabla_Y X + 2Ch(X, Y),$$

for any $X, Y \in \Gamma(TM)$.

Proof. Taking $X, Y \in \Gamma(TM)$ and using (1.4), (1.5), (1.7) and (1.8) we obtain

$$\begin{aligned}
 (\bar{\nabla}_X \Phi)Y &= \bar{\nabla}_X \Phi Y - \Phi \bar{\nabla}_X Y \\
 (2.4) \quad &= \nabla_X \psi Y + h(X, \psi Y) - A_{\omega_Y} X + \nabla_X^\perp \omega Y - \psi \nabla_X Y \\
 &\quad - \omega \nabla_X Y - Bh(X, Y) - Ch(X, Y).
 \end{aligned}$$

Replacing X and Y we get

$$\begin{aligned}
 (\bar{\nabla}_Y \Phi)X &= \nabla_Y \psi X + h(Y, \psi X) - A_{\omega_X} Y + \nabla_Y^\perp \omega X - \psi \nabla_Y X \\
 &\quad - \omega \nabla_Y X - Bh(Y, X) - Ch(Y, X).
 \end{aligned}$$

By using (2.4), (2.5), (1.2) and (1.4) we have

$$\begin{aligned}
 (2.5) \quad 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y \\
 &= \nabla_X \psi Y + h(X, \psi Y) - A_{\omega_Y} X + \nabla_X^\perp \omega Y - \\
 &\quad - \psi \nabla_X Y - \omega \nabla_X Y - Bh(X, Y) - Ch(X, Y) + \nabla_Y \psi X + h(Y, \psi X) \\
 &\quad - A_{\omega_X} Y + \nabla_Y^\perp \omega X - \psi \nabla_Y X - \omega \nabla_Y X - Bh(Y, X) - Ch(Y, X).
 \end{aligned}$$

By comparing to tangent part and normal part of (2.6) we obtain (2.1), (2.2) and (2.3). $\square$

Lemma 2.2. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$. Then*

$$(2.6) \quad \psi \nabla_X \xi - P \nabla_\xi \Phi X + \psi \nabla_\xi X = X,$$

$$(2.7) \quad 2Bh(X, \xi) = Q \nabla_\xi \Phi X,$$

$$(2.8) \quad \omega \nabla_X \xi + \omega \nabla_\xi X + 2Ch(X, \xi) = h(\xi, \Phi X),$$

for any $X \in \Gamma(D)$.

Proof. By using (1.1) and (1.3) we obtain

$$(2.9) \quad (\bar{\nabla}_X \Phi)Y + (\bar{\nabla}_Y \Phi)X = -X$$

for any $X \in \Gamma(D)$. On the other hand, by using (1.4), (1.6), (1.8), (1.10), (1.11) and (1.12) we have

$$\begin{aligned}
 (2.10) \quad (\bar{\nabla}_X \Phi)Y + (\bar{\nabla}_Y \Phi)X &= -\psi \nabla_X \xi - \omega \nabla_X \xi - 2Bh(X, \xi) - 2Ch(X, \xi) \\
 &\quad + P \nabla_\xi \Phi X + Q \nabla_\xi \Phi X + h(\xi, \Phi X) - \psi \nabla_\xi X - \omega \nabla_\xi X.
 \end{aligned}$$

From (2.10) and (2.11) we get

$$\begin{aligned}
 (2.11) \quad -X &= -\psi \nabla_X \xi - \omega \nabla_X \xi - 2Bh(X, \xi) \\
 &\quad - 2Ch(X, \xi) + P \nabla_\xi \Phi X + Q \nabla_\xi \Phi X + h(\xi, \Phi X) - \psi \nabla_\xi X - \omega \nabla_\xi X.
 \end{aligned}$$

By comparing to tangent part and normal part of (2.12) we obtain (2.7), (2.8) and (2.9). $\square$

Lemma 2.3. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. If*

$$(2.12) \quad h(X, \xi) \in \Gamma(\nu),$$

for any $X \in \Gamma(D)$, then

$$(2.13) \quad \nabla_\xi X \in \Gamma(D \oplus \{\xi\}),$$

$$(2.14) \quad \nabla_X \xi \in \Gamma(D \oplus \{\xi\}).$$

Proof. For any $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$, by using (1.1), (1.2), (1.12), (2.8) and (2.13) we obtain

$$(2.15) \quad \begin{aligned} g(\nabla_\xi \Phi X, Z) &= g(Q\nabla_\xi \Phi X, Z) = 2g(Bh(X, \xi), Z) \\ &= 2g(\Phi^2 h(X, \xi), \Phi Z) = -2g(h(X, \xi), \Phi Z) = 0. \end{aligned}$$

From (2.16) we get (2.14).

On the other hand, by using (1.1), (1.2), (1.12), (2.9), (2.13) and (2.14) we have

$$(2.16) \quad g(\nabla_X \xi, Z) = g(\Phi \nabla_X \xi, \Phi Z) = g(\omega \nabla_X \xi, \Phi Z) = 0,$$

From (2.17) we obtain (2.15). $\square$

Lemma 2.4 (see [6]). *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then the distribution $D^\perp$ is integrable if and only if*

$$(2.17) \quad g(A_{\Phi U} V - A_{\Phi V} U - 2\Phi \nabla_U V, \Phi Z) = \eta([U, V])\eta(Z),$$

for any $U, V \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Theorem 2.1. *Let M be a semi-invariant submanifold a nearly Sasakian manifold $\overline{M}$, and let condition (2.13) and condition (2.18) be satisfied. If*

$$(2.18) \quad \nabla_Y X \in \Gamma(D \oplus \{\xi\})$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$, then M is a semi-invariant product of nearly Sasakian manifold $\overline{M}$.

Proof. For any $X, Y \in \Gamma(D)$, from (2.13), (2.14), (2.15) and (2.19) we have

$$(2.19) \quad [X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(D \oplus \{\xi\})$$

$$(2.20) \quad [X, \xi] = \nabla_X \xi - \nabla_\xi X \in \Gamma(D \oplus \{\xi\})$$

From (2.20) and (2.21) we obtain

$$(2.21) \quad [X, Y] \in \Gamma(D \oplus \{\xi\})$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D \oplus \xi)$.

On the other hand, thinking about $[\xi, \xi] = 0$ and (2.21), we have

$$(2.22) \quad [Y, \xi] \in \Gamma(D \oplus \{\xi\})$$

for any $Y \in \Gamma(D \oplus \xi)$.

From (2.22) and (2.23) we get that $[X, Y] \in \Gamma(D \oplus \{\xi\})$, for any $X, Y \in \Gamma(D \oplus \xi)$. Hence the distribution $D \oplus \{\xi\}$ is integrable. Moreover, if M_1 is a leaf of $(D \oplus \{\xi\})$, then from (2.19) and the Gauss formula for the immersion of M_1 in M it follows that M_1 is totally geodesic in M . For any $Y \in \Gamma(TM)$, $X \in \Gamma(D)$ and $U \in \Gamma(D^\perp)$, from (2.19) we have

$$(2.23) \quad g(\nabla_Y U, X) = -g(U, \nabla_Y X) = 0.$$

From (2.24) it follows that $\nabla_Y U \in \Gamma(D^\perp)$ for any $Y \in \Gamma(TM)$ and $U \in \Gamma(D^\perp)$. By using the equation of Gauss for a leaf M_2 of $D^\perp$ we obtain that M_2 is totally geodesic in M . The proof is complete. $\square$

In the references [1], if M is a semi-invariant submanifold of a Sasakian manifold $\overline{M}$, then the distribution $D^\perp$ is integrable and the condition (2.13) is automatically satisfied (p.102). So we have

Corollary 2.1 ([1]). *Let M be a semi-invariant submanifold a Sasakian manifold $\overline{M}$. If the condition (2.19) holds, then M is a semi-invariant product of Sasakian manifold $\overline{M}$.*

Theorem 2.2. *Let M be a semi-invariant submanifold a nearly Sasakian manifold $\overline{M}$, and let condition (2.13) and condition (2.18) be satisfied. If*

$$(2.24) \quad 2Bh(X, Y) = \nabla_X \psi Y - A_{\omega Y} X$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$, then M is a semi-invariant product of nearly Sasakian manifold $\overline{M}$.

Proof. From (2.1) we have

$$(2.25) \quad \begin{aligned} \nabla_Y \Phi X &= 2g(X, Y)\xi - \eta(Y)X + \psi \nabla_X Y \\ &\quad + \psi \nabla_Y X + 2Bh(X, Y) - \nabla_X \psi Y + A_{\omega Y} X, \end{aligned}$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$. Clearly, (2.25) and (2.26) imply that (2.19) holds. Thus, the proof is complete. $\square$

Theorem 2.3. *Let M be a semi-invariant submanifold a nearly Sasakian manifold $\overline{M}$, and let condition (2.13) and condition (2.18) be satisfied. If*

$$(2.26) \quad h(X, \psi Y) + \nabla_X^\perp \omega Y + h(Y, \Phi X) = \omega \nabla_X Y + 2Ch(X, Y),$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$, then M is a semi-invariant product of nearly Sasakian manifold $\overline{M}$.

Proof. From (2.3) we have

$$(2.27) \quad \omega \nabla_Y X = h(X, \psi Y) + \nabla_X^\perp \omega Y + h(Y, \Phi X) - \omega \nabla_X Y - 2Ch(X, Y),$$

for any $Y \in \Gamma(TM)$ and $X \in \Gamma(D)$. Clearly, (2.27) and (2.28) imply that (2.19) holds. Thus, the proof is complete. $\square$

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