

CHENG-MORDESON $\mathcal{L}$ -FUZZY NORMED SPACES AND APPLICATION IN STABILITY OF FUNCTIONAL EQUATION

R. SAADATI AND Y. J. CHO

ABSTRACT. In this paper, we define and study Cheng-Mordeson $\mathcal{L}$ -fuzzy normed spaces. Further, we consider the finite dimensional Cheng-Mordeson $\mathcal{L}$ -fuzzy normed spaces and prove some theorems about completeness, compactness and weak convergence in these spaces. As application, we get a stability result in the setting of Cheng-Mordeson $\mathcal{L}$ -fuzzy normed spaces.

1. INTRODUCTION AND PRELIMINARIES

The theory of fuzzy sets was introduced by Zadeh in 1965 [44]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [2, 21, 15, 16, 18, 19, 20, 29, 39]. One of the problems in $\mathcal{L}$ -fuzzy topology is to obtain an appropriate concept fuzzy normed spaces. In 1984, Katsaras [26] defined a fuzzy norm on a linear space and at the same year Wu and Fang [42] also introduced fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. Some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [8, 9, 14, 28, 40, 43]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [27]. In 2003, Bag and Samanta [6] modified the definition of Cheng and Mordeson [10] by removing a regular condition.

In this paper, we define the notion of Cheng-Mordeson $\mathcal{L}$ -fuzzy normed spaces using [37]. Further, we consider finite dimensional Cheng-Mordeson

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$\mathcal{L}$ -fuzzy normed spaces and prove some theorems about completeness, compactness and weak convergence in these spaces.

In this paper, $\mathcal{L} = (L, \geq_L)$ is a complete lattice, i.e. a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$.

Definition 1.1 (see [17]). 1.1 Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a non-empty set called the universe. An $\mathcal{L}$ -fuzzy set in U is defined as a mapping $\mathcal{A}: U \rightarrow L$. For each u in U , $\mathcal{A}(u)$ represents the *degree* (in L) to which u is an element of $\mathcal{A}$.

Lemma 1.2 (see [12]). Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 1.3 (see [4]). An intuitionistic fuzzy set $\mathcal{A}_{\zeta, \eta}$ in the universe U is an object $\mathcal{A}_{\zeta, \eta} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ for all $u \in U$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta, \eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

We define mapping $\wedge: L^2 \rightarrow L$ as

$$\wedge(x, y) = \begin{cases} x, & \text{if } x \leq_L y \\ y, & \text{if } y \leq_L x \end{cases}.$$

For example,

$$\wedge(x, y) = (\min(x_1, y_1), \max(x_2, y_2)),$$

in which $x = (x_1, x_2), y = (y_1, y_2) \in L^*$.

Definition 1.4. A negator on $\mathcal{L}$ is any decreasing mapping $\mathcal{N}: L \rightarrow L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involution negator*.

The negator N_s on $([0, 1], \leq)$ defined as $N_s(x) = 1 - x$ for all $x \in [0, 1]$ is called the *standard negator* on $([0, 1], \leq)$. In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 1.5. The pair $(V, \mathcal{P})$ is said to be an *Cheng-Mordeson $\mathcal{L}$ -fuzzy normed space* (briefly, *CM $\mathcal{L}$ -fuzzy normed space*) if V is vector space and $\mathcal{P}$ is an $\mathcal{L}$ -fuzzy set on $V \times]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0, +\infty[$,

- (a) $\mathcal{P}(x, t) = 0_{\mathcal{L}}$ for all $t \leq 0$;
- (b) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
- (c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
- (d) $\wedge(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)$;
- (e) $\mathcal{P}(x, \cdot) :]0, \infty[\rightarrow L$ is continuous;

(f) $\lim_{t \rightarrow 0} \mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t \rightarrow \infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.

In this case $\mathcal{P}$ is called an $\mathcal{L}$ -fuzzy norm. If $\mathcal{P} = \mathcal{P}_{\mu, \nu}$ is an intuitionistic fuzzy set (see Definition 1.3), then the pair $(V, \mathcal{P}_{\mu, \nu})$ is said to be an *Cheng-Mordeson intuitionistic fuzzy normed space*.

Example 1.6. Let $(V, \|\cdot\|)$ be a normed space. We define $\wedge(a, b)$ by $\wedge(a, b) := (\min(a_1, b_1), \max(a_2, b_2))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{P}_{\mu, \nu}$ be the intuitionistic fuzzy set on $V \times]0, +\infty[$ defined as follows:

$$\mathcal{P}_{\mu, \nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all $t \in \mathbb{R}^+$. Then $(V, \mathcal{P}_{\mu, \nu})$ is a Cheng-Mordeson intuitionistic fuzzy normed space.

Definition 1.7. (1) A sequence $(x_n)_{n \in \mathbb{N}}$ in a $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$ is called a *Cauchy sequence* if, for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon),$$

where $\mathcal{N}$ is a negator on $\mathcal{L}$.

(2) A sequence $(x_n)_{n \in \mathbb{N}}$ is said to be *convergent* to $x \in V$ in the $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$, which is denoted by $x_n \xrightarrow{\mathcal{P}} x$ if $\mathcal{P}(x_n - x, t) \rightarrow 1_{\mathcal{L}}$, whenever $n \rightarrow +\infty$ for all $t > 0$.

(3) A $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$ is said to be *complete* if and only if every Cauchy sequence in V is convergent.

Lemma 1.8 (see [37]). *Let $\mathcal{P}$ be a $CM\mathcal{L}$ -fuzzy norm on V . Then we have the following:*

- (i) $\mathcal{P}(x, t)$ is nondecreasing with respect to t for all $x \in V$;
- (ii) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in V$ and $t \in]0, +\infty[$.

Definition 1.9. Let $(V, \mathcal{P})$ be an $CM\mathcal{L}$ -fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. For all $t \in]0, +\infty[$, we define the *open ball* $B(x, r, t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as follows:

$$B(x, r, t) = \{y \in V \mid \mathcal{P}(x - y, t) >_L \mathcal{N}(r)\}$$

and define the *unit ball* of V by

$$B(0, r, 1) = \{x : \mathcal{P}(x, 1) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq V$ is said to be *open* if, for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{P}}$ denote the family of all open subsets of V . Then $\tau_{\mathcal{P}}$ is called the *topology induced by the $CM\mathcal{L}$ -fuzzy norm $\mathcal{P}$* .

Definition 1.10. Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. A subset A of V is said to be $\mathcal{LF}$ -*bounded* if there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{P}(x, t) >_L \mathcal{N}(r)$ for all $x \in A$.

Theorem 1.11. *In a $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$, every compact set is closed and $\mathcal{LF}$ -bounded.*

Lemma 1.12 (see [13]). *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space. Let $\mathcal{N}$ be a continuous negator on $\mathcal{L}$. If we define $E_{\lambda, \mathcal{P}}: V \rightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda, \mathcal{P}}(x) = \inf\{t > 0 : \mathcal{P}(x, t) >_L \mathcal{N}(\lambda)\}$$

for all $\lambda \in L \setminus \{0_L, 1_L\}$ and $x \in V$. Then we have the following:

- (i) $E_{\lambda, \mathcal{P}}(\alpha x) = |\alpha| E_{\lambda, \mathcal{P}}(x)$ for all $x \in A$ and $\alpha \in \mathbb{R}$.
- (ii) $E_{\lambda, \mathcal{P}}(x + y) \leq E_{\lambda, \mathcal{P}}(x) + E_{\lambda, \mathcal{P}}(y)$ for all $x, y \in V$.
- (iii) *A sequence $(x_n)_{n \in \mathbb{N}}$ is convergent with respect to the $CM\mathcal{L}$ -fuzzy norm $\mathcal{P}$ if and only if $E_{\lambda, \mathcal{P}}(x_n - x) \rightarrow 0$. Also, the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the $CM\mathcal{L}$ -fuzzy norm $\mathcal{P}$ if and only if it is a Cauchy sequence with respect to $E_{\lambda, \mathcal{P}}$.*

Lemma 1.13 (see [13]). *A subset A of $\mathbb{R}$ is $\mathcal{LF}$ -bounded in $(\mathbb{R}, \mathcal{P})$ if and only if it is bounded in $\mathbb{R}$.*

Corollary 1.14 (see [13]). *If the real sequence $(\beta_n)_{n \in \mathbb{N}}$ is $\mathcal{LF}$ -bounded, then it has at least one limit point.*

Definition 1.15. Let V be a vector space and let f be a real functional on V . We define

$$\tilde{V} = \{f : \mathcal{P}_0(f(x), t) \geq_L \mathcal{P}(cx, t), c \neq 0\}$$

for all $t > 0$.

Lemma 1.16 (see [38]). *If $(V, \mathcal{P})$ is a $CM\mathcal{L}$ -fuzzy normed space, then we have*

- (a) *the function $(x, y) \rightarrow x + y$ is continuous.*
- (b) *the function $(\alpha, x) \rightarrow \alpha x$ is continuous.*

By the above lemma, a $CM\mathcal{L}$ -fuzzy normed space is Hausdorff Topological Vector Space.

2. $CM\mathcal{L}$ -FUZZY FINITE DIMENSIONAL NORMED SPACES

Theorem 2.1. *Let $\{x_1, \dots, x_n\}$ be a linearly independent set of vectors in vector space V and let $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space. Then there exist $c \neq 0$ and a $CM\mathcal{L}$ -fuzzy normed space $(\mathbb{R}, \mathcal{P}_0)$ such that, for every choice of the n real scalars $\alpha_1, \dots, \alpha_n$,*

$$(2.1) \quad \mathcal{P}(\alpha_1 x_1 + \dots + \alpha_n x_n, t) \leq_L \mathcal{P}_0\left(c \sum_{j=1}^n |\alpha_j|, t\right).$$

Proof. Put $s = |\alpha_1| + \dots + |\alpha_n|$. If $s = 0$, all α_j 's must be zero and so (2.1) holds for any c . Let $s > 0$. Then (2.1) is equivalent to the inequality which we

obtain from (2.1) by dividing by s and putting $\beta_j = \frac{\alpha_j}{s}$, that is,

$$(2.2) \quad \mathcal{P}(\beta_1 x_1 + \cdots + \beta_n x_n, t') \leq \mathcal{P}_0(c, t'), \quad (t' = \frac{t}{s} \sum_{j=1}^n |\beta_j| = 1).$$

Hence, it suffices to prove the existence of a $c \neq 0$ and $\mathcal{L}$ -fuzzy norm $\mathcal{P}_0$ such that (2.2) holds. Suppose that this is not true. Then there exists a sequence $(y_m)_{m \in \mathbb{N}}$ of vectors,

$$y_m = \beta_{1,m} x_1 + \cdots + \beta_{n,m} x_n, \quad \left(\sum_{j=1}^n |\beta_{j,m}| = 1 \right)$$

such that $\mathcal{P}(y_m, t) \rightarrow 1_{\mathcal{L}}$ as $m \rightarrow \infty$ for all $t > 0$. Since $\sum_{j=1}^n |\beta_{j,m}| = 1$, we have $|\beta_{j,m}| \leq 1$ and so, by Lemma 1.13, the sequence of $(\beta_{j,m})$ is $\mathcal{LF}$ -bounded. By Corollary 1.14, $(\beta_{1,m})$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and let $(y_{1,m})$ denote the corresponding subsequence of (y_m) . By the same argument, $(y_{1,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence $\beta_2^{(m)}$ of real scalars convergence. Let β_2 denote the limit. Continuing this process, after n steps, we obtain a subsequence $(y_{n,m})_m$ of (y_m) such that

$$y_{n,m} = \sum_{j=1}^n \gamma_{j,m} x_j \left(\sum_{j=1}^n |\gamma_{j,m}| = 1 \right)$$

and $\gamma_{j,m} \rightarrow \beta_j$ as $m \rightarrow \infty$. By Lemma 1.12 (ii), for any $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, we have

$$\begin{aligned} E_{\mu, \mathcal{P}}(y_{n,m} - \sum_{j=1}^n \beta_j x_j) &= E_{\mu, \mathcal{P}}(\sum_{j=1}^n (\gamma_{j,m} - \beta_j) x_j) \\ &\leq \sum_{j=1}^n |\gamma_{j,m} - \beta_j| E_{\mu, \mathcal{P}}(x_j) \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. By Lemma 1.12 (iii), we conclude

$$\lim_{m \rightarrow \infty} y_{n,m} = \sum_{j=1}^n \beta_j x_j \left(\sum_{j=1}^n |\beta_j| = 1 \right),$$

so that not all β_j can be zero. Put $y = \sum_{j=1}^n \beta_j x_j$. Since $\{x_1, \dots, x_n\}$ is a linearly independent set, we have $y \neq 0$. Since $\mathcal{P}(y_m, t) \rightarrow 1_{\mathcal{L}}$ by assumption, we have $\mathcal{P}(y_{n,m}, t) \rightarrow 1_{\mathcal{L}}$. Hence it follows that

$$\mathcal{P}(y, t) = \mathcal{P}((y - y_{n,m}) + y_{n,m}, t) \geq_L \wedge(\mathcal{P}(y - y_{n,m}, t/2), \mathcal{P}(y_{n,m}, t/2)) \rightarrow 1_{\mathcal{L}}$$

and so $y = 0$, which is a contradiction. $\square$

Theorem 2.2. *Every finite dimensional subspace W of a $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$ is complete. In particular, every finite dimensional $CM\mathcal{L}$ -fuzzy normed space is complete.*

Proof. Let $(y_m)_{m \in \mathbb{N}}$ be a Cauchy sequence in W such that y is its limit. Then we show that $y \in W$. Let $\dim W = n$ and $\{x_1, \dots, x_n\}$ any linearly independent subset for W . Then each y_m has a unique representation of the form

$$y_m = \alpha_1^{(m)}x_1 + \dots + \alpha_n^{(m)}x_n.$$

Since $(y_m)_{m \in \mathbb{N}}$ is a Cauchy sequence, for any $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there is a positive integer n_0 such that

$$\mathcal{N}(\varepsilon) <_L \mathcal{P}(y_m - y_k, t),$$

whenever $m, k > n_0$ and $t > 0$. From this and the last theorem, we have

$$\begin{aligned} \mathcal{N}(\varepsilon) <_L \mathcal{P}(y_m - y_k, t) &= \mathcal{P}\left(\sum_{j=1}^n (\alpha_j^{(m)} - \alpha_j^{(k)})x_j, t\right) \\ &\leq_L \mathcal{P}_0\left(\sum_{j=1}^n |\alpha_j^{(m)} - \alpha_j^{(k)}|c, t\right) \leq_L \mathcal{P}_0\left(1, \frac{\frac{t}{c}}{\sum_{j=1}^n |\alpha_j^{(m)} - \alpha_j^{(k)}|}\right) \\ &\leq_L \mathcal{P}_0\left(1, \frac{\frac{t}{c}}{|\alpha_j^{(m)} - \alpha_j^{(k)}|}\right) = \mathcal{P}_0\left(\alpha_j^{(m)} - \alpha_j^{(k)}, \frac{t}{c}\right) \end{aligned}$$

for some $c \neq 0$ and $\mathcal{P}_0$. This shows that each of the n sequences $(\alpha_j^{(m)})_{m \in \mathbb{N}}$ where $j \in \{1, 2, 3, \dots, n\}$ is a Cauchy sequence in $\mathbb{R}$. Hence these sequences converge. Let α_j denote the limit. Using these n limits $\alpha_1, \dots, \alpha_n$, we define

$$y = \alpha_1x_1 + \dots + \alpha_nx_n.$$

Clearly, $y \in W$. Furthermore, by Lemma 1.12 (ii), for any $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, we have

$$E_{\mu, \mathcal{P}}(y_m - y) = E_{\mu, \mathcal{P}}\left(\sum_{j=1}^n ((\alpha_j^{(m)} - \alpha_j)x_j)\right) \leq \sum_{j=1}^n |\alpha_j^{(m)} - \alpha_j| E_{\mu, \mathcal{P}}(x_j) \rightarrow 0$$

whenever $m \rightarrow \infty$. This shows that the arbitrary sequence $(y_m)_{m \in \mathbb{N}}$ is convergent in W . Hence W is complete. $\square$

Corollary 2.3. *Every finite dimensional subspace W of a $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$ is closed in V .*

Theorem 2.4. *In a finite dimensional $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$, any subset $K \subset V$ is compact if and only if K is closed and $\mathcal{LF}$ -bounded.*

Proof. By Theorem 1.11, compactness implies closedness and $\mathcal{LF}$ -boundedness.

Conversely, let K be closed and $\mathcal{LF}$ -bounded. Let $\dim V = n$ and $\{x_1, \dots, x_n\}$ be a linearly independent set of V . We consider a sequence $(x^{(m)})_{m \in \mathbb{N}}$ in

K . Each $x^{(m)}$ has a representation by

$$x^{(m)} = \alpha_1^{(m)}x_1 + \cdots + \alpha_n^{(m)}x_n.$$

Since, K is $\mathcal{LF}$ -bounded, so is $(x^{(m)})_{m \in \mathbb{N}}$ and so there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{P}(x^{(m)}, t) >_L \mathcal{N}(r)$ for all $m \in \mathbb{N}$.

On the other hand, by Theorem 2.1, there exist $c \neq 0$ and a $\mathcal{L}$ -fuzzy norm $\mathcal{P}_0$ such that

$$\begin{aligned} \mathcal{N}(r) <_L \mathcal{P}(x^{(m)}, t) &= \mathcal{P}\left(\sum_{j=1}^n \alpha_j^{(m)}x_j, t\right) \\ &\leq_L \mathcal{P}_0\left(c \sum_{j=1}^n |\alpha_j^{(m)}|, t\right) \leq_L \mathcal{P}_0\left(1, \frac{t}{c \sum_{j=1}^n |\alpha_j^{(m)}|}\right) \\ &\leq_L \mathcal{P}_0\left(1, \frac{t}{c|\alpha_j^{(m)}|}\right) = \mathcal{P}_0\left(\alpha_j^{(m)}, \frac{t}{c}\right). \end{aligned}$$

Hence, the sequence $(\alpha_j^{(m)})_{m \in \mathbb{N}}$ for any fixed j is $\mathcal{LF}$ -bounded and, by Corollary 1.14, has a limit point α_j , where $1 \leq j \leq n$. We consider that $(x^{(m)})_{m \in \mathbb{N}}$ has a subsequence $(z_m)_{m \in \mathbb{N}}$ which converges to $z = \sum_{j=1}^n \alpha_j x_j$. Since K is closed, $z \in K$. This shows that an arbitrary sequence $(x^{(m)})_{m \in \mathbb{N}}$ in K has a subsequence which converges in K . Hence, K is compact. $\square$

Remark 2.5. In a $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$ whenever $\mathcal{P}(x, t) >_L \mathcal{N}(r)$ for all $x \in V$, $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, we can find $t_0 \in]0, t[$ such that $\mathcal{P}(x, t_0) >_L \mathcal{N}(r)$ (see [15]).

Lemma 2.6. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space and let A be a subspace of V . Define*

$$\mathcal{D}(x_1 - A, t) = \sup\{\mathcal{P}(x_1 - y, t) : y \in A\}$$

for all $x_1 \in V$ and $t > 0$. Then, for any $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$ and $x_1 \in V \setminus A$, there exists $y_1 \in A$ such that

$$\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{P}(x_1 - y_1, t) \leq_L \mathcal{D}(x_1 - A, t).$$

The proof is straightforward.

Lemma 2.7. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space and let A be a subset of V . If we define*

$$p_1 = \inf\{t > 0 : \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda)\}$$

and

$$p_2 = \inf\{t > 0 : \wedge(\mathcal{D}(x_1 - A, t), \varepsilon) >_L \mathcal{N}(\lambda)\},$$

in which $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$. Then there exists $\delta \in]0, t[$ such that $p_2 \geq p_1 + \delta$.

Proof. Since $\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{D}(x_1 - A, t)$, by Remark 2.5, there exists $\delta \in]0, t[$ such that $\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{D}(x_1 - A, t - \delta)$ and so

$$\begin{aligned} p_2 &= \inf\{t > 0 : \wedge(\mathcal{D}(x_1 - A, t), \varepsilon) >_L \mathcal{N}(\lambda)\} \\ &\geq \inf\{t > 0 : \mathcal{D}(x_1 - A, t - \delta) >_L \mathcal{N}(\lambda)\} \\ &= \inf\{t + \delta > 0 : \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda)\} = p_1 + \delta. \end{aligned} \quad \square$$

Lemma 2.8. *Let $(V, \mathcal{P})$ be a CML -fuzzy normed space and let A be a nonempty closed subspace of V . Then $x \in A$ if and only if $\mathcal{D}(x - A, t) = 1_{\mathcal{L}}$ for all $t > 0$.*

Proof. Let $\mathcal{D}(x - A, t) = 1_{\mathcal{L}}$. By definition, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that $\mathcal{P}(x - x_n, t) \rightarrow 1_{\mathcal{L}}$. Hence $x - x_n \rightarrow 0$ or equivalently $x_n \rightarrow x$ and, since A is closed, $x \in A$. The converse is trivial. $\square$

Theorem 2.9. *Let $(V, \mathcal{P})$ be a CML -fuzzy normed space and let A be a nonempty closed subspace of V . Then, for any $y \in A$, there exist $x_0 \in V \setminus A$ and $\lambda_0 \in L$ such that $x_0 \in B(0, \lambda, 1)$ and $E_{\lambda, \mathcal{P}}(x_0 - y) \geq 1$ for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$.*

Proof. Since, A is a nonempty closed subspace of V , by Lemma 2.8, there exists $x_1 \in V \setminus A$ such that $\mathcal{D}(x_1 - A, t) <_L 1_{\mathcal{L}}$ for all $t > 0$. Let

$$\sup_{t > 0} \mathcal{D}(x_1 - A, t) = \sigma.$$

Let $\lambda_0 = \mathcal{N}(\sigma)$. Then, for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$, we have

$$\sup_{t > 0} \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda).$$

By the property of sup, there exists $t_0 > 0$ such that $\mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda)$ for all $t \geq t_0$. By Lemma 2.6, there exists $y_1 \in A$ such that

$$\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{P}(x_1 - y_1, t)$$

for all $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$ and $t \geq 0$. Taking $x_0 = \frac{x_1 - y_1}{p_2}$, by Lemma 2.7, we have

$$\begin{aligned} \mathcal{P}(x_0, 1) &= \mathcal{P}\left(\frac{x_1 - y_1}{p_2}, 1\right) = \mathcal{P}(x_1 - y_1, p_2) \geq_L \wedge(\mathcal{D}(x_1 - A, p_2), \varepsilon) \\ &\geq_L \wedge(\mathcal{D}(x_1 - A, p_1 + \delta), \varepsilon) >_L \wedge(\mathcal{N}(\lambda), \varepsilon). \end{aligned}$$

Since, $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$ is arbitrary, we have $\mathcal{P}(x_0, 1) >_L \mathcal{N}(\lambda)$, i.e., $x_0 \in B(0, \lambda, 1)$ for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$. Taking $\delta_1 = \frac{\delta}{p_2}$, by Lemma 2.7, we have

$$\begin{aligned} \wedge(\mathcal{P}(x_0 - y, N_s(\delta_1)), \varepsilon) &= \wedge(\mathcal{P}(x_1 - (y_1 + p_2 y), p_2 N_s(\delta_1)), \varepsilon) \\ &\leq_L \wedge(\mathcal{D}(x_1 - A, p_2 - \delta), \varepsilon) \leq_L \mathcal{N}(\lambda). \end{aligned}$$

Letting $\varepsilon \rightarrow 1_{\mathcal{L}}$ and $\delta \rightarrow 0$, we have $\mathcal{P}(x_0 - y, 1) \leq_L \mathcal{N}(\lambda)$ and so

$$E_{\lambda, \mathcal{P}}(x_0 - y) \geq 1$$

for all $y \in A$ and $x_0 \in B(0, \lambda, 1)$. $\square$

Lemma 2.10. *Let $\{x_1, \dots, x_n\}$ be a linearly independent set of vectors in vector space V and $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space. Then there exists $k \neq 0$ such that, for every choice of the n real scalars $\alpha_1, \dots, \alpha_n$,*

$$E_{\lambda, \mathcal{P}}\left(\sum_{j=1}^n \alpha_j x_j\right) \geq |k| \sum_{j=1}^n |\alpha_j|.$$

Proof. By Theorem 2.1, there exist $c \neq 0$ and an $\mathcal{L}$ -fuzzy norm $\mathcal{P}_0$ such that

$$\mathcal{P}\left(\sum_{j=1}^n \alpha_j x_j, t\right) \leq_L \mathcal{P}_0\left(c \sum_{j=1}^n |\alpha_j|, t\right).$$

Therefore, we have

$$E_{\lambda, \mathcal{P}}\left(\sum_{j=1}^n \alpha_j x_j\right) \geq E_{\lambda, \mathcal{P}_0}\left(c \sum_{j=1}^n |\alpha_j|\right) = |c| \sum_{j=1}^n |\alpha_j| E_{\lambda, \mathcal{P}_0}(1).$$

Taking $k = cE_{\lambda, \mathcal{P}_0}(1)$, we have

$$E_{\lambda, \mathcal{P}}\left(\sum_{j=1}^n \alpha_j x_j\right) \geq |k| \sum_{j=1}^n |\alpha_j|. \quad \square$$

Theorem 2.11. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space. Then $(V, \mathcal{P})$ is finite dimensional if and only if the unit ball $B(0, \lambda, 1)$ is compact.*

Proof. Let $\dim V = n$ and $\{x_1, \dots, x_n\}$ a basis for V . We consider any sequence $(x^{(m)})_{m \in \mathbb{N}}$ in $B(0, \lambda, 1)$. Each $x^{(m)}$ has the representation by

$$x^{(m)} = \sum_{j=1}^n \alpha_j^{(m)} x_j.$$

By Lemmas 2.7 and 2.10, we have

$$1 \geq E_{\lambda, \mathcal{P}}(x^{(m)}) \geq |k| \sum_{j=1}^n |\alpha_j^{(m)}|,$$

where $k \neq 0$. Hence the sequence $(\alpha_j^{(m)})_{m \in \mathbb{N}}$ is bounded and has a limit point α_j ($1 \leq j \leq n$). Therefore, $(x^{(m)})_{m \in \mathbb{N}}$ has a subsequence $(x^{(m_k)})_{k \in \mathbb{N}}$ which converges to $x = \sum_{j=1}^n \alpha_j x_j$.

On the other hand, for any $\varepsilon \neq 0_{\mathcal{L}}$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\mathcal{P}(x, 1 + \delta) \geq_L \wedge(\mathcal{P}(x^{(m_k)} - x, \delta), \mathcal{P}(x^{(m_k)}, 1)) \geq_L \wedge(\mathcal{N}(\varepsilon), \mathcal{N}(\lambda))$$

for all $\delta > 0$. Since $\varepsilon \neq_L 0_{\mathcal{L}}$ and $\delta > 0$ are arbitrary, it follows that

$$\mathcal{P}(x, 1) \geq_L \wedge(1_{\mathcal{L}}, \mathcal{N}(\lambda)) = \mathcal{N}(\lambda)$$

and, consequently, $x \in B(0, \lambda, 1)$. Hence, $B(0, \lambda, 1)$ is compact.

Conversely, assume that the unit balls be compact, but $(V, \mathcal{P})$ is not finite dimensional. We choose $x_1 \neq 0$ in V , for any $k_1 \in \mathbb{R}$, let $V_1 = \{k_1 x_1 : x_1 \in$

$V, k_1 \in \mathbb{R}\}$. By Theorem 2.9, for all $\lambda_{0,1} <_L \lambda \leq_L 1_{\mathcal{L}}$, there exist $x_2 \in V \setminus V_1$ and $x_2 \in B(0, \lambda, 1)$ such that $E_{\lambda, \mathcal{P}}(x_2 - x_1) \geq 1$.

In this case, x_1 and x_2 are linear independent. In fact, if x_1 and x_2 are dependent, then there exists $k_1, k_2 \in \mathbb{R}$ (we might as well assume $k_2 \neq 0$) such that $k_1 x_1 + k_2 x_2 = 0$ and $x_2 = \frac{-k_1}{k_2} x_1 \in V_1$, which is a contradiction.

Let $V_2 = \{k_1 x_1 + k_2 x_2 : x_1 \in V_1, x_2 \in V \setminus V_1, k_1, k_2 \in \mathbb{R}\}$. By Theorem 2.9, for all $\lambda_{0,2} <_L \lambda \leq_L 1_{\mathcal{L}}$, there exist $x_3 \in V \setminus V_2$ and $x_3 \in B(0, \lambda, 1)$ such that $E_{\lambda, \mathcal{P}}(x_3 - y) \geq 1$ where $y \in V_2$. In particular, if we choose $y = x_1$ and $y = x_2$, then $E_{\lambda, \mathcal{P}}(x_3 - x_1) \geq 1$ and $E_{\lambda, \mathcal{P}}(x_3 - x_2) \geq 1$. By the same way, we can choose $(x_n)_{n \in \mathbb{N}} \subset B(0, \lambda, 1)$ such that $E_{\lambda, \mathcal{P}}(x_m - x_n) \geq 1$ where $m \neq n$ for all $\lambda_{0,n-1} <_L \lambda \leq_L 1_{\mathcal{L}}$. If we put $\lambda_0 = \bigvee_{1 \leq i \leq n-1} \lambda_{0,i}$, then the sequence $(x_n)_{n \geq 2}$ lie in $B(0, \lambda, 1)$ and $E_{\lambda, \mathcal{P}}(x_m - x_n) \geq 1$ for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$. By Lemma 1.12, (ii), the sequence $(x_n)_{n \geq 2}$ has not any convergent subsequence in V , which is a contradiction. This completes the proof. $\square$

Theorem 2.12. *Let $(V, \mathcal{P})$ be a finite dimensional $CM\mathcal{L}$ -fuzzy normed space and let A be a closed subspace of V . Then, for all $\lambda >_L \lambda_0$, there exists $x_0 \in B(0, \lambda, 1)$ such that*

$$\inf_{y \in A} E_{\lambda, \mathcal{P}}(x_0 - y) = 1.$$

Proof. By Theorem 2.9, for any $y_n \in A$, there exist $x_n \in V \setminus A$ and $\lambda_0 \in L$ such that

$$(2.3) \quad x_n \in B(0, \lambda, 1), \quad E_{\lambda, \mathcal{P}}(x_n - y_n) \geq 1$$

for all $\lambda >_L \lambda_0$. Since V is finite dimensional, by Theorem 2.11, $B(0, \lambda, 1)$ is compact and so there exists $x_0 \in B(0, \lambda, 1)$ such that

$$\mathcal{P}(x_{n_k} - x_0, t) \rightarrow 1_{\mathcal{L}}$$

for all $t > 0$, where $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $(x_n)_{n \in \mathbb{N}}$. Since $x_0 \in B(0, \lambda, 1)$, $E_{\lambda, \mathcal{P}}(x_0) \leq 1$. Since the null element $0 \in A$, we have

$$1 \geq E_{\lambda, \mathcal{P}}(x_0) = E_{\lambda, \mathcal{P}}(x_0 - 0) \geq \inf_{y \in A} E_{\lambda, \mathcal{P}}(x_0 - y).$$

Next, we prove that $\inf_{y \in A} E_{\lambda, \mathcal{P}}(x_0 - y) \geq 1$. By (2.1), $\mathcal{P}(x_n - y_n, 1) \leq_L \mathcal{N}(\lambda)$. Let $\mathcal{P}(x_0 - y, 1) >_L \mathcal{N}(\lambda)$ for all $y \in A$. Then, by continuity of $CM\mathcal{L}$ -fuzzy norm $\mathcal{P}$ and Remark 2.5, we can find $\lambda_1 \in L$ such that, for $\delta \in]0, 1[$,

$$\mathcal{P}(x_0 - y, N_s(\delta)) >_L \mathcal{N}(\lambda_1),$$

and

$$\mathcal{N}(\lambda_1) >_L \mathcal{N}(\lambda).$$

Since $x_{n_k} \rightarrow x_0$, there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$,

$$\mathcal{P}(x_{n_k} - x_0, t) >_L \mathcal{N}(\lambda_1)$$

for all $t > 0$. By triangle inequality 1.5, (d), we have

$$\begin{aligned}\mathcal{N}(\lambda) &\geq_L \mathcal{P}(x_{n_k} - y_{n_k}, t) \geq_L \wedge(\mathcal{P}(x_{n_k} - x_0, t/2), \mathcal{P}(x_0 - y_{n_k}, t/2)) \\ &\geq_L \wedge(\mathcal{N}(\lambda_1), \mathcal{N}(\lambda_1)) >_L \mathcal{N}(\lambda),\end{aligned}$$

which is a contradiction. Then, for any $y \in A$, we have $\mathcal{P}(x_0 - y, 1) \leq_L \mathcal{N}(\lambda)$, which implies $\inf_{y \in A} E_{\lambda, \mathcal{P}}(x_0 - y) \geq 1$. This completes the proof. $\square$

Definition 2.13. A sequence $(x_m)_{m \in \mathbb{N}}$ in a $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$ is said to be *weakly convergent* if there exists $x \in V$ such that, for all $f \in \tilde{V}$ and $t > 0$,

$$\mathcal{P}(f(x_m) - f(x), t) \rightarrow 1_{\mathcal{L}}.$$

This is written by

$$x_m \xrightarrow{W} x.$$

Theorem 2.14. Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$ -fuzzy normed space and let $(x_m)_{m \in \mathbb{N}}$ be a sequence in V . Then we have the following:

- (i) Convergence implies weak convergence with the same limit.
- (ii) If $\dim V < \infty$, then weak convergence implies convergence.

Proof. (i) Let $x_m \rightarrow x$. Then, for all $t > 0$, we have

$$\mathcal{P}(x_m - x, t) \rightarrow 1_{\mathcal{L}}.$$

By Definition 1.15, for every $f \in \tilde{V}$,

$$\mathcal{P}_0(f(x_m) - f(x), t) = \mathcal{P}_0(f(x_m - x), t) \geq_L \mathcal{P}(x_m - x, t/c)(c \neq 0).$$

Then $x_m \xrightarrow{W} x$.

(ii) Let $x_m \xrightarrow{W} x$ and $\dim V = n$. Let $\{x_1, \dots, x_n\}$ be a linearly independent set of V . Then $x_m = \alpha_1^{(m)}x_1 + \dots + \alpha_n^{(m)}x_n$ and $x = \alpha_1x_1 + \dots + \alpha_nx_n$. By assumption, for all $f \in \tilde{V}$ and $t > 0$, we have

$$\mathcal{P}_0(f(x_m) - f(x), t) \rightarrow 1_{\mathcal{L}}.$$

We take in particular $f_1, \dots, f_n$ defined by $f_jx_j = 1$ and $f_jx_i = 0$ ($i \neq j$). Therefore, $f_j(x_m) = \alpha_j^{(m)}$ and $f_j(x) = \alpha_j$. Hence $f_j(x_m) \rightarrow f_j(x)$ implies $\alpha_j^{(m)} \rightarrow \alpha_j$. From this and Lemma 1.12 (ii), we obtain

$$E_{\mu, \mathcal{P}}(x_m - x) = E_{\mu, \mathcal{P}}\left(\sum_{j=1}^n (\alpha_j^{(m)} - \alpha_j)x_j\right) \leq \sum_{j=1}^n |\alpha_j^{(m)} - \alpha_j| E_{\lambda, \mathcal{P}}(x_j) \rightarrow 0$$

as $m \rightarrow \infty$. This shows that $(x_m)_{m \in \mathbb{N}}$ convergence to x . $\square$

Theorem 2.15. A $CM\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P})$ is locally convex.

Proof. It suffices to consider the family of neighborhoods of the origin, $B(0, r, t)$, with $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$. Let $t > 0$, $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, $x, y \in B(0, r, t)$ and $\alpha \in [0, 1]$. Then we have

$$\begin{aligned} \mathcal{P}(\alpha x + (1 - \alpha)y, t) &\geq_L \wedge(\mathcal{P}(\alpha x, \alpha t), \mathcal{P}((1 - \alpha)y, (1 - \alpha)t)) \\ &= \wedge(\mathcal{P}(x, t), \mathcal{P}(y, t)) >_L \mathcal{N}(r). \end{aligned}$$

Thus, $\alpha x + (1 - \alpha)y$ belongs to $B(0, r, t)$ for all $\alpha \in [0, 1]$. □

3. STABILITY OF CUBIC FUNCTIONAL EQUATIONS IN $\mathcal{L}$ -FUZZY NORMED SPACES

The study of stability problems for functional equations is related to a question of Ulam [41] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [22]. Subsequently, the result of Hyers was generalized by T. Aoki [3] for additive mappings and by Th.M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. The paper [34] of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. We refer the interested readers for more information on such problems to e.g. [5, 11, 23, 35, 36].

The functional equation

$$(3.1) \quad 3f(x + 3y) + f(3x - y) = 15f(x + y) + 15f(x - y) + 80f(y)$$

is said to be the cubic functional equation since the function $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability problem for the cubic functional equation was proved by Jun and Kim [24] for mappings $f: X \rightarrow Y$, where X is a real normed space and Y is a Banach space. Later a number of mathematicians worked on the stability of some types of the cubic equation [25, 34]. In addition, Mirmostafaei, Mirzavaziri and Moslehian [33, 32], Alsina [1], Miheţ and Radu [30], Miheţ et. al. [31] and Baktash et. al. [7] investigated the stability in the settings of fuzzy, probabilistic and random normed spaces.

The aim of this note, is to provide a result on the stability of the cubic functional equation (3.1) in fuzzy normed spaces and give a better error estimation.

Now, we state our main result.

Theorem 3.1. *Let X be a linear space, $(Z, \mathcal{P}')$ be a CML -fuzzy normed space, $\varphi: X \times X \rightarrow Z$ be a function such that for some $0 < \alpha < 27$,*

$$(3.2) \quad \mathcal{P}'(\varphi(3x, 0), t) \geq_L \mathcal{P}'(\alpha\varphi(x, 0), t) \quad (x, y \in X, t > 0)$$

and $\lim_{n \rightarrow \infty} \mathcal{P}'(\varphi(3^n x, 3^n y), 27^n t) = 1_L$ for all $x, y \in X$ and $t > 0$. Let $(Y, \mathcal{P})$ be a complete fuzzy normed space. If $f: X \rightarrow Y$ is a mapping such that

$$(3.3) \quad \mathcal{P}(3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y), t) \geq_L \mathcal{P}'(\varphi(x, y), t)$$

where $x, y \in X, t > 0$. Then there exists a unique cubic mapping $C: X \rightarrow Y$ such that

$$(3.4) \quad \mathcal{P}(f(x) - C(x), t) \geq_L \mathcal{P}'(\varphi(x, 0), (27 - \alpha)t).$$

Proof. Putting $y = 0$ in (3.3) we get

$$(3.5) \quad \mathcal{P}\left(\frac{f(3x)}{27} - f(x), t\right) \geq_L \mathcal{P}(\varphi(x, 0), 27t) \quad (x \in X, t > 0).$$

Replacing x by $3^n x$ in (3.5), and using (3.2) we obtain

$$(3.6) \quad \mathcal{P}\left(\frac{f(3^{n+1}x)}{27^{n+1}} - \frac{f(3^n x)}{27^n}, t\right) \geq_L \mathcal{P}'(\varphi(3^n x, 0), 27 \times 27^n t) \geq_L \mathcal{P}'(\varphi(x, 0), \frac{27 \times 27^n}{\alpha^n} t).$$

Since $\frac{f(3^n x)}{27^n} - f(x) = \sum_{k=0}^{n-1} \left(\frac{f(3^{k+1}x)}{27^{k+1}} - \frac{f(3^k x)}{27^k} \right)$, by (3.6) we have

$$\mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t \sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}\right) \geq_L \bigwedge_{k=0}^{n-1} \mathcal{P}'(\varphi(x, 0), t) = \mathcal{P}'(\varphi(x, 0), t),$$

that is,

$$(3.7) \quad \mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t\right) \geq_L \mathcal{P}'\left(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}}\right).$$

By replacing x with $3^m x$ in (3.7) we observe that:

$$(3.8) \quad \mathcal{P}\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, t\right) \geq_L \mathcal{P}'\left(\varphi(x, 0), \frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{27 \times 27^k}}\right).$$

Then $\{\frac{f(3^n x)}{27^n}\}$ is a Cauchy sequence in $(Y, \mathcal{P})$. Since $(Y, \mathcal{P})$ is a complete $CM\mathcal{L}$ -fuzzy normed space this sequence convergent to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (3.8) to obtain

$$(3.9) \quad \mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t\right) \geq_L \mathcal{P}'\left(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}}\right),$$

and so for every $\delta > 0$ we have

$$(3.10) \quad \begin{aligned} \mathcal{P}(C(x) - f(x), t + \delta) &\geq_L \wedge \left(\mathcal{P}(C(x) - \frac{f(3^n x)}{27^n}, \delta), \mathcal{P}(\frac{f(3^n x)}{27^n} - f(x), t) \right) \\ &\geq_L \wedge \left(\mathcal{P}\left(C(x) - \frac{f(3^n x)}{27^n}, \delta\right), \mathcal{P}'\left(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}}\right) \right). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (3.10) we get

$$(3.11) \quad \mathcal{P}(C(x) - f(x), t + \delta) \geq_L \mathcal{P}'(\varphi(x, 0), t(27 - \alpha)).$$

Since δ was arbitrary, by taking $\delta \rightarrow 0$ in (3.11) we get

$$\mathcal{P}(C(x) - f(x), t) \geq_L \mathcal{P}'(\varphi(x, 0), t(27 - \alpha)).$$

Replacing x, y by $3^n x, 3^n y$ in (3.3) to get

$$\begin{aligned} \mathcal{P}\left(\frac{f(3^n(x+3y))}{27^n} + \frac{f(3^n(3x-y))}{27^n} - \frac{15f(3^n(x+y))}{27^n} - \frac{15f(3^n(x-y))}{8^n} - \right. \\ \left. - \frac{80f(3^n(y))}{27^n}, t\right) \geq_L \mathcal{P}'(\varphi(3^n x, 3^n y), 27^n t), \end{aligned}$$

for all $x, y \in X$ and for all $t > 0$. Since $\lim_{n \rightarrow \infty} \mathcal{P}'(\varphi(3^n x, 3^n y), 27^n t) = 1$ we conclude that C fulfills (3.1). To Prove the uniqueness of the cubic function C , assume that there exists a cubic function $D: X \rightarrow Y$ which satisfies (3.4). Fix $x \in X$. Clearly $C(3^n x) = 27^n C(x)$ and $D(3^n x) = 27^n D(x)$ for all $n \in \mathbb{N}$. It follows from (3.4) that

$$\begin{aligned} \mathcal{P}(C(x) - D(x), t) &= \mathcal{P}\left(\frac{C(3^n x)}{27^n} - \frac{D(3^n x)}{27^n}, t\right) \\ &\geq_L \wedge \left(\mathcal{P}\left(\frac{C(3^n x)}{27^n} - \frac{f(3^n x)}{27^n}, \frac{t}{2}\right), \mathcal{P}\left(\frac{D(3^n x)}{27^n} - \frac{f(3^n x)}{27^n}, \frac{t}{2}\right) \right) \\ &\geq_L \mathcal{P}'\left(\varphi(3^n x, 0), 27^n(27 - \alpha)\frac{t}{2}\right) \geq_L \mathcal{P}'\left(\varphi(x, 0), \frac{27^n(27 - \alpha)\frac{t}{2}}{\alpha^n}\right). \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \frac{27^n(27 - \alpha)t}{2\alpha^n} = \infty,$$

we get

$$\lim_{n \rightarrow \infty} \mathcal{P}'(\varphi(x, 0), \frac{27^n(27 - \alpha)t}{2\alpha^n}) = 1_L.$$

Therefore $\mathcal{P}(C(x) - D(x), t) = 1_L$ for all $t > 0$, whence $C(x) = D(x)$. $\square$

Corollary 3.2. *Let X be a linear space, $L = [0, 1]$, $(Z, \mathcal{P}')$ be a $CM\mathcal{L}$ -fuzzy normed space, $(Y, \mathcal{P})$ be a complete $CM\mathcal{L}$ -fuzzy normed space, p, q be nonnegative real numbers and let $z_0 \in Z$. If $f: X \rightarrow Y$ is a mapping such that*

$$(3.12) \quad \mathcal{P}(3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y), t) \\ \geq \mathcal{P}'((\|x\|^p + \|y\|^q)z_0, t) \quad (x, y \in X, t > 0),$$

$f(0) = 0$ and $p, q < 3$, then there exists a unique cubic mapping $C: X \rightarrow Y$ such that

$$(3.13) \quad \mathcal{P}(f(x) - C(x), t) \geq \mathcal{P}'(\|x\|^p z_0, (27 - 3^p)t).$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi: X \times X \rightarrow Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 3^p$. $\square$

Corollary 3.3. *Let X be a linear space, $L = [0, 1]$, $(Z, \mathcal{P}')$ be a $CM\mathcal{L}$ -fuzzy normed space, $(Y, \mathcal{P})$ be a complete $CM\mathcal{L}$ -fuzzy normed space and let $z_0 \in Z$. If $f: X \rightarrow Y$ is a mapping such that*

$$(3.14) \quad \mathcal{P}(3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y), t) \geq \mathcal{P}'(\varepsilon z_0, t)$$

for $x, y \in X, t > 0$ and $f(0) = 0$, then there exists a unique cubic mapping $C: X \rightarrow Y$ such that

$$(3.15) \quad \mathcal{P}(f(x) - C(x), t) \geq \mathcal{P}'(\varepsilon z_0, 26t).$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi: X \times X \rightarrow Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 1$. $\square$

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REZA SAADATI,
DEPARTMENT OF MATHEMATICS,
SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY,
TEHRAN, IRAN
E-mail address: rsaadati@eml.cc

YEOL JE CHO,
DEPARTMENT OF MATHEMATICS EDUCATION AND THE RINS,
GYEONGSANG NATIONAL UNIVERSITY,
CHINJU 660-701, KOREA.
E-mail address: yjcho@gnu.ac.kr