

**SOME APPLICATIONS OF DIFFERENTIAL
SUBORDINATION OF MULTIVALENT FUNCTIONS
ASSOCIATED WITH THE WRIGHT GENERALIZED
HYPERGEOMETRIC FUNCTION**

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ABSTRACT. Making use of the principle of differential subordination, we investigate some inclusion relationships of certain subclasses of multivalent analytic functions associated with the Wright generalized hypergeometric function.

1. INTRODUCTION

Let $A_n(p)$ denote the class of functions of the form:

$$(1.1) \quad f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (p, n \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic and p -valent in the open unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For convenience, we write $A_1(p) = A(p)$. If f and g are analytic in U , we say that f is subordinate to g , written symbolically as follows: $f \prec g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $f(z) = g(w(z))$ ($z \in U$). In particular, if the function g is univalent in U , then we have the following equivalence (cf. [2, 14], see also [15, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f \in A_n(p)$, given by (1.1), and $g \in A_n(p)$ given by

$$(1.2) \quad g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (p, n \in \mathbb{N}),$$

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then the Hadamard product (or convolution) of f and g is defined by

$$(1.3) \quad (f * g)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z) \quad (p, n \in \mathbb{N}; z \in U).$$

Let $\alpha_1, A_1, \dots, \alpha_q, A_q$ and $\beta_1, B_1, \dots, \beta_s, B_s$ ($q, s \in \mathbb{N}$) be positive real parameters such that

$$1 + \sum_{i=1}^s B_i - \sum_{i=1}^q A_i \geq 0.$$

The Wright generalized hypergeometric function [31] (see also [28])

$$\begin{aligned} {}_q\Psi_s [(\alpha_1, A_1), \dots, (\alpha_q, A_q); (\beta_1, B_1), \dots, (\beta_s, B_s); z] \\ = {}_q\Psi_s [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] \end{aligned}$$

is defined by

$${}_q\Psi_s [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^q \Gamma(\alpha_i + nA_i)}{\prod_{i=1}^s \Gamma(\beta_i + nB_i)} \cdot \frac{z^n}{n!} \quad (z \in U).$$

If $A_i = 1$ ($i = 1, \dots, q$) and $B_i = 1$ ($i = 1, \dots, s$), we have the relationship:

$$\Omega {}_q\Psi_s [(\alpha_i, 1)_{1,q}; (\beta_i, 1)_{1,s}; z] = {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z),$$

where ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ is the generalized hypergeometric function (see [28]) and

$$(1.4) \quad \Omega = \frac{\prod_{i=1}^s \Gamma(\beta_i)}{\prod_{i=1}^q \Gamma(\alpha_i)}.$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [23, 24]).

By using the generalized hypergeometric function Dziok and Srivastava [7] introduced a linear operator. In [8] Dziok and Raina and in [1] Aouf and Dziok extended the linear operator by using the Wright generalized hypergeometric function.

First we define a function ${}_q\Phi_s^p [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$ by

$${}_q\Phi_s^p [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \Omega z^p {}_q\Psi_s [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$$

and consider the following linear operator

$$\theta_p [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}] : A_n(p) \rightarrow A_n(p),$$

defined by the convolution

$$\theta_p [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}] f(z) = {}_q\Phi_s^p [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] * f(z).$$

We observe that, for a function f of the form (1.1), we have

$$(1.5) \quad \theta_p [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}] f(z) = z^p + \sum_{k=n}^{\infty} \Omega \sigma_n(\alpha_1) a_{k+p} z^{k+p},$$

where Ω is given by (1.4) and $\sigma_n(\alpha_1)$ is defined by

$$(1.6) \quad \sigma_n(\alpha_1) = \frac{[\Gamma(\alpha_1 + A_1 n) \dots \Gamma(\alpha_q + A_q n)]}{\Gamma(\beta_1 + B_1 n) \dots \Gamma(\beta_s + B_s n) n!}.$$

If, for convenience, we write

$$\theta_{p,q,s} [\alpha_1, A_1, B_1] = \theta_p [(\alpha_1, A_1), \dots, (\alpha_q, A_q); (\beta_1, B_1), \dots, (\beta_s, B_s)] f(z),$$

then one can easily verify from the definition (1.5) that

$$(1.7) \quad \begin{aligned} z A_1 (\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))' &= \alpha_1 \theta_{p,q,s} [\alpha_1 + 1, A_1, B_1] f(z) \\ &- (\alpha_1 - p A_1) \theta_{p,q,s} [\alpha_1, A_1, B_1] f(z) \quad (A_1 > 0). \end{aligned}$$

The linear operator $\theta_{1,q,s} [\alpha_1, A_1, B_1] = \theta [\alpha_1]$ was introduced by Dziok and Raina [8] and studied by Aouf and Dziok [1].

We note that, for $f \in A_n(p)$, $A_i = 1$ ($i = 1, \dots, q$), $B_i = 1$ ($i = 1, \dots, s$) and by specializing the parameters α_i ($i = 1, \dots, q$), β_i ($i = 1, \dots, s$), q and s we obtain the following operators studied by various authors:

- (i) $\theta_{p,q,s} [\alpha_1] f(z) = H_{p,q,s}(\alpha_1) f(z)$ (see Patel et al. [22]);
- (ii) $\theta_{p,2,1} [a, 1; c] f(z) = L_p(a, c) f(z)$ ($a > 0, c > 0$) (see Carlson and Shaffer [3] and Saitoh [25]);
- (iii) $\theta_{p,2,1} [\mu + p, 1; 1] f(z) = D^{\mu+p-1} f(z)$ ($\mu > -p$), where $D^{\mu+p-1} f(z)$ is the $(\mu + p - 1)$ -th order Ruscheweyh derivative of a function $f \in A_n(p)$ (see Kumar and Shukla [10]);
- (iv) $\theta_{p,2,1} [1 + p, 1; 1 + p - \mu] f(z) = \Omega_z^{(\mu,p)} f(z)$, where the operator $\Omega_z^{(\mu,p)} f(z)$ is defined by (see Srivastava and Aouf [27])

$$\Omega_z^{(\mu,p)} f(z) = \frac{\Gamma(1 + p - \mu)}{\Gamma(1 + p)} z^\mu D_z^\mu f(z) \quad (0 \leq \mu < 1; p \in \mathbb{N}),$$

where $\Omega_z^\mu f(z)$ is the fractional derivative operator (see, for details, [6] and [18] and [19]);

- (v) $\theta_{p,2,1} [\delta + p, 1; \delta + p + 1] f(z) = F_{\delta,p}(f)(z)$, where $F_{\delta,p}(f)$ is the generalized Bernardi-Libera-Livingston operator (see [5]), defined by

$$F_{\delta,p}(f)(z) = \frac{\delta + p}{z^\delta} \int_0^z t^{\delta-1} f(t) dt \quad (\delta > -p; p \in \mathbb{N});$$

- (vi) $\theta_{p,2,1} [p + 1, 1; m + p] f(z) = I_{m,p} f(z)$ ($m \in \mathbb{Z}; m > -p$), where the operator $I_{m,p}$ is the $(m + p - 1)$ -th Noor operator, considered by Liu and Noor [12];
- (vii) $\theta_{p,2,1} [\lambda + p, c; a] f(z) = I_p^\lambda(a, c) f(z)$ ($a, c \in \mathbb{R} \setminus \mathbb{N}_0^-; \lambda > -p$), where $I_p^\lambda(a, c) f(z)$ is the Cho-Kwon-Srivastava operator (see [4]).

For fixed parameters A and B ($-1 \leq B < A \leq 1$), we say that a function $f \in A_n(p)$ is in the class $Q_{p,q,s}^n(\alpha_1, A_1, B_1; A, B)$, if it satisfies the following subordination condition:

$$(1.8) \quad \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{pz^{p-1}} \prec \frac{1 + Az}{1 + Bz} \quad (p \in \mathbb{N}).$$

In view of the definition of subordination, (1.8) is equivalent to the following condition:

$$(1.9) \quad \left| \frac{\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{z^{p-1}} - p}{B \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{z^{p-1}} - pA} \right| < 1 \quad (z \in U).$$

For convenience, we write $Q_{p,q,s}^n(\alpha_1, A_1, B; 1 - \frac{2\theta}{p}, -1) = Q_{p,q,s}^n(\alpha_1, A_1, B_1; \theta)$, where $Q_{p,q,s}^n(\alpha_1, A_1, B_1; \theta)$ denote the class of functions in $A_n(p)$ satisfying the following inequality:

$$(1.10) \quad \operatorname{Re} \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{z^{p-1}} > \theta \quad (0 \leq \theta < p; p \in \mathbb{N}; z \in U).$$

2. PRELIMINARIES

To establish our main results, we shall need the following lemmas.

Lemma 1 ([9]). *Let the function h be analytic and convex (univalent) in U with $h(0) = 1$. Suppose also the function φ given by*

$$(2.1) \quad \varphi(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \dots,$$

is analytic in U . If

$$(2.2) \quad \varphi(z) + \frac{1}{\gamma} z \varphi'(z) \prec h(z),$$

where $\gamma \neq 0$ and $\operatorname{Re} \gamma \geq 0$. Then

$$(2.3) \quad \varphi(z) \prec \Psi(z) = \frac{\gamma}{n} z^{-\frac{\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t) dt \prec h(z),$$

and Ψ is the best dominant of (2.2).

With a view to stating a well-known result (Lemma 2 below), we denote by $P(\delta)$ the class of functions Φ given by

$$(2.4) \quad \Phi(z) = 1 + c_1 z + c_2 z^2 + \dots,$$

which are analytic in U and satisfy the following inequality:

$$\operatorname{Re} \Phi(z) > \delta \quad (0 \leq \delta < 1; z \in U).$$

Lemma 2 ([20]). *Let the function Φ , given by (2.4), be in the class $P(\delta)$. Then*

$$\operatorname{Re} \Phi(z) \geq 2\delta - 1 + \frac{2(1-\delta)}{1+|z|} \quad (0 \leq \delta < 1; z \in U).$$

Lemma 3 ([29]). *For $0 \leq \gamma_1, \gamma_2 < 1$,*

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3) \quad (\gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2)).$$

The result is the best possible.

Lemma 4 ([26]). *Let Φ be analytic in U with*

$$\Phi(0) = 1 \text{ and } \operatorname{Re} \Phi(z) > \frac{1}{2} \quad (z \in U).$$

*Then, for any function F analytic in U , $(\Phi * F)(U)$ is contained in the convex hull of $F(U)$.*

Lemma 5 ([17]). *Let φ be analytic in U with $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for $0 < |z| < 1$, and let $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$.*

(i) Let $B \neq 0$ and $\gamma \in \mathbb{C}^ = \mathbb{C} \setminus \{0\}$ satisfy either*

$$\left| \frac{\gamma(A-B)}{B} - 1 \right| \leq 1 \text{ or } \left| \frac{\gamma(A-B)}{B} + 1 \right| \leq 1.$$

If φ satisfies

$$1 + \frac{z\varphi'(z)}{\gamma\varphi(z)} \prec \frac{1 + Az}{1 + Bz},$$

then

$$\varphi(z) \prec (1 + Bz)^{\gamma \left(\frac{A-B}{B} \right)}$$

and this is the best dominant.

(ii) Let $B = 0$ and $\gamma \in \mathbb{C}^$ be such that $|\gamma A| < \pi$. If φ satisfies*

$$1 + \frac{z\varphi'(z)}{\gamma\varphi(z)} \prec 1 + Az,$$

then

$$\varphi(z) \prec e^{\gamma Az}$$

and this is the best dominant.

For real or complex numbers a, b and $c (c \notin \mathbb{Z}_0^-)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots.$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details, [30, Chapter 14]).

Each of the identities (asserted by Lemma 6 below) is well-known (cf., e.g., [30, Chapter 14]).

Lemma 6 ([30]). *For real or complex parameters a, b and $c (c \notin \mathbb{Z}_0^-)$,*

$$(2.5) \quad \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \\ = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z) \quad (\operatorname{Re}(c) > \operatorname{Re}(b) > 0);$$

$$(2.6) \quad {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z);$$

and

$$(2.7) \quad {}_2F_1(a, b; c; z) = (1-z)_2^{-a} {}_2F_1(a, c-b; c; \frac{z}{z-1}).$$

3. MAIN RESULTS

Unless otherwise mentioned we shall assume through this paper that $-1 \leq B < A \leq 1$, $\lambda, A_1 > 0$ and $p, n \in \mathbb{N}$.

Theorem 1. *Let the function f defined by (1.1) satisfy the following subordination condition*

$$(3.1) \quad (1-\lambda) \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} + \lambda \frac{(\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z))'}{pz^{p-1}} \\ \prec \frac{1 + Az}{1 + Bz}.$$

Then

$$(3.2) \quad \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} \prec Q(z) \prec \frac{1 + Az}{1 + Bz},$$

where the function Q given by

$$(3.3) \quad Q(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)_2^{-1} F_1(1, 1; \frac{\alpha_1}{A_1 \lambda n} + 1; \frac{Bz}{1 + Bz}) & (B \neq 0), \\ 1 + \frac{A\alpha_1}{A_1 \lambda n + \alpha_1} z & (B = 0), \end{cases}$$

is the best dominant of (3.2). Furthermore

$$(3.4) \quad \operatorname{Re} \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} > \rho \quad (z \in U),$$

where

$$(3.5) \quad \rho(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)_2^{-1} F_1(1, 1; \frac{\alpha_1}{A_1 \lambda n} + 1; \frac{B}{B-1}) & (B \neq 0), \\ 1 - \frac{A\alpha_1}{A_1 \lambda n + \alpha_1} & (B = 0), \end{cases}$$

the estimate in (3.4) is the best possible.

Proof. Consider the function φ defined by

$$(3.6) \quad \varphi(z) = \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{pz^{p-1}} \quad (z \in U).$$

Then φ is of the form (2.1) and is analytic in U . Applying the identity (1.7) in (3.6) and differentiating the resulting equation with respect to z , we get

$$\begin{aligned} (1-\lambda) \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{pz^{p-1}} + \lambda \frac{(\theta_{p,q,s}[\alpha_1+1, A_1, B_1]f(z))'}{pz^{p-1}} \\ = \varphi(z) + \frac{A_1\lambda}{\alpha_1} z \varphi'(z) \prec \frac{1+Az}{1+Bz}. \end{aligned}$$

Now, by using Lemma 1 for $\gamma = \frac{\alpha_1}{A_1\lambda}$, we obtain

$$\begin{aligned} \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{pz^{p-1}} \prec Q(z) \prec \frac{\alpha_1}{A_1\lambda n} z^{\frac{-\alpha_1}{A_1\lambda n}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda n}-1} \left(\frac{1+At}{1+Bt} \right) dt \\ = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1-Bz)^{-1} F_1(1, 1; \frac{\alpha_1}{A_1\lambda n} + 1; \frac{Bz}{1+Bz}) & (B \neq 0), \\ 1 + \frac{\alpha_1 A}{A_1\lambda n + \alpha_1} z & (B = 0), \end{cases} \end{aligned}$$

by change of variables followed by use of the identities (2.5), (2.6) and (2.7) (with $a = 1, c = b + 1, b = \frac{\alpha_1}{A_1\lambda n}$). This proves the assertion (3.2) of Theorem 1.

Next, in order to prove the assertion (3.4) of Theorem 1, it suffices to show that

$$(3.7) \quad \inf_{|z|<1} \{\operatorname{Re} Q(z)\} = Q(-1).$$

Indeed we have, for $|z| \leq r < 1$,

$$\operatorname{Re} \frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br}.$$

Upon setting

$$g(\zeta, z) = \frac{1+A\zeta z}{1+B\zeta z} \text{ and } dv(\zeta) = \frac{\alpha_1}{A_1\lambda n} \zeta^{\frac{\alpha_1}{A_1\lambda n}-1} d\zeta \quad (0 \leq \zeta \leq 1),$$

which is a positive measure on the closed interval $[0, 1]$, we get

$$Q(z) = \int_0^1 g(\zeta, z) dv(\zeta),$$

so that

$$\operatorname{Re} Q(z) \geq \int_0^1 \left(\frac{1 - A\zeta r}{1 - B\zeta r} \right) dv(\zeta) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \rightarrow 1^-$ in the above inequalities, we obtain the assertion (3.4) of Theorem 1. Finally, the estimate in (3.4) is the best possible as the function Q is the best dominant of (3.2). $\square$

Taking $\lambda = 1$, $A = 1 - \frac{2\sigma}{p}$ ($0 \leq \sigma < p$) and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *The following inclusion property holds true for the class $\mathcal{Q}_{p,q,s}^n(\alpha_1, A_1, B_1; \theta)$:*

$$\begin{aligned} \mathcal{Q}_{p,q,s}^n(\alpha_1 + 1, A_1, B_1; \theta) &\subset \mathcal{Q}_{p,q,s}^n(\alpha_1, A_1, B_1; \beta(p, n, \alpha_1, A_1, \theta)) \\ &\subset \mathcal{Q}_{p,q,s}^n(\alpha_1, A_1, B_1; \theta), \end{aligned}$$

where

$$\beta(p, n, \alpha_1, A_1, \theta) = \theta + (p - \theta) \left\{ {}_2F_1\left(1, 1; \frac{\alpha_1}{A_1 n} + 1; \frac{1}{2}\right) - 1 \right\}.$$

The result is the best possible.

Taking $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *The following inclusion property holds true for the function class $\mathcal{Q}_{p,q,s}^n(\alpha_1, A_1, B_1; A, B)$:*

$$\begin{aligned} \mathcal{Q}_{p,q,s}^n(\alpha_1 + 1, A_1, B_1; A, B) &\subset \mathcal{Q}_{p,q,s}^n(\alpha_1, A_1, B_1; 1 - \frac{2\theta}{p}, -1) \\ &\subset \mathcal{Q}_{p,q,s}^n(\alpha_1, A_1, B_1; A, B), \quad 0 \leq \theta < p, \end{aligned}$$

where

$$\theta = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)_2^{-1} {}_2F_1\left(1, 1; \frac{\alpha_1}{A_1 n} + 1; \frac{B}{B-1}\right) & (B \neq 0), \\ 1 - \frac{\alpha_1 A}{A_1 n + \alpha_1} & (B = 0). \end{cases}$$

The result is the best possible.

Theorem 2. *If $f \in \mathcal{Q}_{p,q,s}^n(\alpha_1, A_1, B_1; \theta)$ ($0 \leq \theta < 1$), then*

$$(3.8) \quad \operatorname{Re} \frac{(1 - \lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z))'}{pz^{p-1}} > \theta(|z| < R),$$

where

$$R = \left\{ \frac{\sqrt{\alpha_1^2 + \lambda^2 A_1^2 n^2} - \lambda A_1 n}{\alpha_1} \right\}^{\frac{1}{n}}.$$

The result is the best possible.

Proof. Since $f \in Q_{p,q,s}^n(\alpha_1, A_1, B_1; \theta)$, we write

$$(3.9) \quad \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1; \theta] f(z))'}{pz^{p-1}} = \theta + (1 - \theta)u(z) \quad (z \in U).$$

Then, clearly, u is of the form (2.1), is analytic in U , and has a positive real part in U . Making use of the identity (1.7) in (3.9) and differentiating the resulting equation with respect to z , we obtain

$$(3.10) \quad \frac{1}{(1 - \theta)} \left\{ \frac{(1 - \lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z))'}{pz^{p-1}} - \theta \right\} \\ = u(z) + \frac{A_1 \lambda}{\alpha_1} z u'(z).$$

Now, by applying the well-known estimate [13]

$$\frac{|zu'(z)|}{\operatorname{Re} u(z)} \leq \frac{2nr^n}{1 - r^{2n}} \quad (|z| = r < 1)$$

in (3.10), we get

$$(3.11) \quad \frac{1}{(1 - \theta)} \operatorname{Re} \left\{ \frac{(1 - \lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z))'}{pz^{p-1}} - \theta \right\} \\ \geq \operatorname{Re} u(z) \left(1 - \frac{2A_1 \lambda n r^n}{\alpha_1 (1 - r^{2n})} \right).$$

It is easily seen that the right-hand side of (3.11) is positive provided that $r < R$, where R is given as in Theorem 2. This proves the assertion (3.8) of Theorem 2.

In order to show that the bound R is the best possible, we consider the function $f \in A_n(p)$ defined by

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} = \theta + (1 - \theta) \frac{1 + z^n}{1 - z^n} \quad (0 \leq \theta < 1; z \in U).$$

Noting that

$$\frac{1}{(1 - \theta)} \left\{ \frac{(1 - \lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z))'}{pz^{p-1}} - \theta \right\} \\ = \frac{\alpha_1 - \alpha_1 z^{2n} - 2A_1 \lambda n z^n}{\alpha_1 (1 - z^n)^2} = 0,$$

for $z = R \exp(\frac{i\pi}{n})$. This completes the proof of Theorem 2. $\square$

Putting $\lambda = 1$ in Theorem 2, we obtain the following result.

Corollary 3. *If $f \in Q_{p,q,s}^n(\alpha_1, A_1, B_1; \theta)$ ($0 \leq \theta < 1$), then $f \in Q_{p,q,s}^n(\alpha_1 + 1, A_1, B_1; \theta)$ for $|z| < \tilde{R}$, where*

$$\tilde{R} = \left\{ \frac{\sqrt{\alpha_1^2 + A_1^2 n^2} - A_1 n}{\alpha_1} \right\}^{\frac{1}{n}}.$$

The result is the best possible.

For a function $f \in A_n(p)$, the generalized Bernardi-Libera-Livingston integral operator $F_{\delta,p}$ is defined by

$$\begin{aligned} F_{\delta,p}(f)(z) &= \frac{\delta + p}{z^p} \int_0^z t^{\delta-1} f(t) dt \\ (3.12) \quad &= \left(z^p + \sum_{k=n}^{\infty} \frac{\delta + p}{\delta + p + k} z^{p+k} \right) * f(z) \quad (\delta > -p) \\ &= z_2^p F_1(1, \delta + p, \delta + p + 1; z) * f(z). \end{aligned}$$

Theorem 3. *Let $f \in Q_{p,q,s}^n(\alpha_1, A_1, B_1; A, B)$ and let the operator $F_{\delta,p}(f)$ defined by (3.12). Then*

$$(3.13) \quad \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \prec \theta(z) \prec \frac{1 + Az}{1 + Bz},$$

where the function θ given by

$$(3.14) \quad \theta(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)_2^{-1} F_1(1, 1; \frac{p+\delta}{n} + 1; \frac{Bz}{Bz+1}) & (B \neq 0), \\ 1 + \frac{(p+\delta)A}{p+\delta+n}z & (B = 0). \end{cases}$$

is the best dominant of (3.13). Furthermore,

$$(3.15) \quad \operatorname{Re} \left\{ \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \right\} > \xi^* \quad (z \in U),$$

where

$$(3.16) \quad \xi^* = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)_2^{-1} F_1(1, 1; \frac{p+\delta}{n} + 1; \frac{B}{B+1}) & (B \neq 0) \\ 1 - \frac{(p+\delta)A}{p+\delta+n} & (B = 0). \end{cases}$$

The result is the best possible.

Proof. From (1.7) and (3.12) it follows that

$$\begin{aligned} z(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))' &= (p + \delta)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)) - \\ (3.17) \quad &\delta(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'. \end{aligned}$$

By setting

$$(3.18) \quad \varphi(z) = \frac{(\theta_{p,q,s} [\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \quad (z \in U),$$

we note that $\varphi(z)$ is of the form (2.1) and is analytic in U . Using the identity (3.17) in (3.18), and then differentiating the resulting equation with respect to z , we obtain

$$\frac{(\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} = \varphi(z) + \frac{z\varphi'(z)}{p+\delta} \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above. $\square$

Remark 1. We observe that.

$$(3.19) \quad \frac{(\theta_{p,q,s} [\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} = \frac{\delta+p}{pz^{\delta+p}} \int_0^z t^\delta (\theta_{p,q,s} [\alpha_1, A_1, B_1] f(t))' dt$$

$(f \in A_n(p); z \in U).$

In view of (3.19), Theorem 3 for $A = 1 - 2\mu$ ($0 \leq \mu < 1$) and $B = -1$ yields the following corollary.

Corollary 4. *If $\delta > 0$ and if $f \in A_n(p)$ satisfies the following inequality*

$$\operatorname{Re} \frac{(\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} > \mu \quad (0 \leq \mu < 1; z \in U),$$

then

$$\operatorname{Re} \frac{\delta+p}{pz^{p+\delta}} \int_0^z t^\delta (\theta_{p,q,s} [\alpha_1, A_1, B_1] f(t))' dt >$$

$$\mu + (1-\mu) \left[{}_2F_1 \left(1, 1; \frac{p+\delta}{n} + 1; \frac{1}{2} \right) - 1 \right] \quad (z \in U).$$

The result is the best possible.

Theorem 4. *Let $f \in A_n(p)$. Suppose also that $g \in A_n(p)$ satisfies the following inequality:*

$$\operatorname{Re} \frac{\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z)}{z^p} > 0 \quad (z \in U).$$

If

$$\left| \frac{\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z)} - 1 \right| < 1 \quad (z \in U),$$

then

$$\operatorname{Re} \frac{z(\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z)} > 0 \quad (|z| < R_0),$$

where

$$R_0 = \frac{\sqrt{9n^2 + 4p(p+n)} - 3n}{2(p+n)}.$$

Proof. Letting

$$(3.20) \quad w(z) = \frac{\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z)} - 1 = k_n z^n + k_{n+1} z^{n+1} + \dots,$$

we note that w is analytic in U , with

$$w(0) = 0 \text{ and } |w(z)| \leq |z|^n \quad (z \in U).$$

Then, by applying the familiar Schwarz lemma [16], we obtain

$$w(z) = z^n \Psi(z),$$

where the function Ψ is analytic in U and $|\Psi(z)| \leq 1$ ($z \in U$). Therefore, (3.20) leads us to

$$(3.21) \quad \theta_{p,q,s} [\alpha_1, A_1, B_1] f(z) = \theta_{p,q,s} [\alpha_1, A_1, B_1] g(z) (1 + z^n \Psi(z)) \quad (z \in U).$$

Differentiating (3.21) logarithmically with respect to z , we obtain

$$(3.22) \quad \frac{z(\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z)} = \frac{z(\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z))'}{\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z)} + \frac{z^n \{n\Psi(z) + z\Psi'(z)\}}{1 + z^n \Psi(z)}.$$

Putting $\varphi(z) = \frac{\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z)}{z^p}$, we see that the function $\varphi(z)$ is of the form (2.1), is analytic in U , $\operatorname{Re} \varphi(z) > 0$ ($z \in U$) and

$$\frac{z(\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z))'}{\theta_{p,q,s} [\alpha_1, A_1, B_1] g(z)} = \frac{z\varphi'(z)}{\varphi(z)} + p,$$

so that we find from (3.22) that

$$(3.23) \quad \operatorname{Re} \frac{(\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z)} \geq p - \left| \frac{z\varphi'(z)}{\varphi(z)} \right| - \left| \frac{z^n \{n\Psi(z) + z\Psi'(z)\}}{1 + z^n \Psi(z)} \right| \quad (z \in U).$$

Now, by using the following known estimates [13] (see also [21]) :

$$\left| \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{2nr^{n-1}}{1-r^{2n}} \text{ and } \left| \frac{n\Psi(z) + z\Psi'(z)}{1 + z^n \Psi(z)} \right| \leq \frac{n}{1-r^n} \quad (|z| = r < 1),$$

in (3.21), we obtain

$$\operatorname{Re} \frac{z(\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z)} \geq \frac{p - 3nr^n - (p+n)r^{2n}}{1-r^{2n}} \quad (|z| = r < 1),$$

which is certainly positive, provided that $r < R_0$, R_0 being given as in Theorem 4. $\square$

Theorem 5. *Let $-1 \leq D_j < C_j \leq 1$ ($j = 1, 2$). If each of the functions $f_j \in A_n(p)$ satisfies the following subordination condition:*

$$(3.24) \quad (1-\lambda) \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z)}{z^p} + \lambda \frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f_j(z)}{z^p} \prec \frac{1 + C_j z}{1 + D_j z},$$

then

$$(3.25) \quad (1-\lambda) \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] G(z)}{z^p} + \lambda \frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] G(z)}{z^p} \prec \frac{1 + (1 - 2\eta)z}{1 - z},$$

where

$$G(z) = \theta_{p,q,s}[\alpha_1, A_1, B_1] (f_1 * f_2)(z)$$

and

$$\eta = 1 - \frac{4(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left[1 - \frac{1}{2} F_1\left(1, 1; \frac{\alpha_1}{A_1 \lambda} + 1; \frac{1}{2}\right) \right].$$

The result is the best possible when $D_1 = D_2 = -1$.

Proof. Suppose that each of the functions $f_j \in A_n(p)$ ($j = 1, 2$) satisfies the condition (3.24). Then, by letting

$$(3.26) \quad \varphi_j(z) = (1-\lambda) \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z)}{z^p} + \lambda \frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f_j(z)}{z^p} \quad (j = 1, 2),$$

we have

$$\varphi_j(z) \in P(\gamma_j) \quad \left(\gamma_j = \frac{1 - C_j}{1 - D_j}; j = 1, 2 \right).$$

By making use of identity (1.7) in (3.26), we observe that

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z) = \frac{\alpha_1}{A_1 \lambda} z^{p - \frac{\alpha_1}{A_1 \lambda}} \int_0^z t^{\frac{\alpha_1}{A_1 \lambda} - 1} \varphi_j(t) dt \quad (j = 1, 2),$$

which in view of the definition of G given already with (3.25) yields

$$(3.27) \quad \theta_{p,q,s}[\alpha_1, A_1, B_1] G(z) = \frac{\alpha_1}{A_1 \lambda} z^{p - \frac{\alpha_1}{A_1 \lambda}} \int_0^z t^{\frac{\alpha_1}{A_1 \lambda} - 1} \varphi_0(t) dt,$$

where, for convenience,

$$(3.28) \quad \begin{aligned} \varphi_0(z) &= (1-\lambda) \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] G(z)}{z^p} + \frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] G(z)}{z^p} \\ &= \frac{\alpha_1}{A_1 \lambda} z^{-\frac{\alpha_1}{A_1 \lambda}} \int_0^z t^{\frac{\alpha_1}{A_1 \lambda} - 1} (\varphi_1 * \varphi_2)(t) dt. \end{aligned}$$

Since $\varphi_1 \in P(\gamma_1)$ and $\varphi_2 \in P(\gamma_2)$, it follows from Lemma 3 that

$$(3.29) \quad (\varphi_1 * \varphi_2)(z) \in P(\gamma_3) \quad (\gamma_3 = 1 - 2(1 - \gamma_1)((1 - \gamma_2))).$$

Now, by using (3.29) in (3.28) and then appealing to Lemma 2 and Lemma 4, we get

$$\begin{aligned} \operatorname{Re} \{\varphi_0(z)\} &= \frac{\alpha_1}{A_1\lambda} \int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1} \operatorname{Re} \{(\varphi_1 * \varphi_2)\}(uz) du \\ &\geq \frac{\alpha_1}{A_1\lambda} \int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1} (2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u|z|}) du \\ &> \frac{\alpha_1}{A_1\lambda} \int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1} (2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u}) du \\ &= 1 - \frac{4(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left(1 - \frac{\alpha_1}{A_1\lambda} \int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1} (1 + u)^{-1} du \right) \\ &= 1 - \frac{4(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left[1 - \frac{1}{2} F_1(1, 1; \frac{\alpha_1}{A_1\lambda} + 1; \frac{1}{2}) \right] \\ &= \eta(z \in U). \end{aligned}$$

When $D_1 = D_2 = -1$, we consider the functions $f_j(z) \in A_n(p)$ ($j = 1, 2$), which satisfy the hypothesis (3.24) of Theorem 5 and are defined by

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z) = \frac{\alpha_1}{A_1\lambda} z^{-\frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda}-1} \left(\frac{1 + C_j t}{1 - t} \right) dt \quad (j = 1, 2).$$

Thus it follows from (3.28) and Lemma 2 that

$$\begin{aligned} \varphi_0(z) &= \frac{\alpha_1}{A_1\lambda} \int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1} \left\{ 1 - (1 + C_1)(1 + C_2) + \frac{(1 + C_1)(1 + C_2)}{1 - uz} \right\} du \\ &= 1 - (1 + C_1)(1 + C_2) + (1 + C_1)(1 + C_2)(1 - z)^{-1} {}_2F_1 \left(1, 1; \frac{\alpha_1}{A_1\lambda} + 1; \frac{z}{z - 1} \right) \\ &\rightarrow 1 - (1 + C_1)(1 + C_2) + \frac{1}{2}(1 + C_1)(1 + C_2) {}_2F_1 \left(1, 1; \frac{\alpha_1}{A_1\lambda} + 1; \frac{1}{2} \right) \end{aligned}$$

as $z \rightarrow 1^-$, which completes the proof of Theorem 5. $\square$

Remark 2. Taking $A_i = 1$ ($i = 1, \dots, q$), $B_i = 1$ ($i = 1, \dots, s$) and $j = 1$ in Theorem 5, we obtain the result obtained by Liu [11, Theorem 2.4].

Putting $A_i = 1 (i = 1, \dots, q)$, $B_i = 1 (i = 1, \dots, s)$, $C_j = 1 - 2\theta_j (0 \leq \theta_j < 1)$, $D_j = 1 (j = 1, 2)$, $q = s + 1$, $\alpha_1 = \beta_1 = p$, $\alpha_j = 1 (j = 2, 3, \dots, s + 1)$ and $\beta_j = 1 (j = 2, 3, \dots, s)$ in Theorem 5, we obtain the following result.

Corollary 5. *If the functions $f_j \in A_n(p) (j = 1, 2)$ satisfy the following inequality:*

$$(3.30) \quad \operatorname{Re} \left\{ (1 - \lambda) \frac{f_j(z)}{z^p} + \lambda \frac{f'_j(z)}{pz^{p-1}} \right\} > \theta_j \quad (0 \leq \theta_j < 1; j = 1, 2; z \in U),$$

then

$$\operatorname{Re} \left\{ (1 - \lambda) \frac{(f_1 * f_2)(z)}{z^p} + \lambda \frac{z(f_1 * f_2)'(z)}{pz^{p-1}} \right\} > \eta_0 \quad (z \in U),$$

where

$$\eta_0 = 1 - 4(1 - \theta_1)(1 - \theta_2) \left[1 - \frac{1}{2_2} F_1(1, 1; \frac{p}{\lambda} + 1; \frac{1}{2}) \right].$$

The result is the best possible.

Theorem 6. *Let the function f be defined by (1.1) be in the class $Q_{p,q,s}^n[\alpha_1, A_1, B_1; A, B]$ and let $g \in A_n(p)$ satisfy the following inequality:*

$$\operatorname{Re} \frac{g(z)}{z^p} > \frac{1}{2} \quad (z \in U).$$

Then

$$(f * g)(z) \in Q_{p,q,s}^n[\alpha_1, A_1, B_1; A, B].$$

Proof. We have

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1](f * g)(z))'}{pz^{p-1}} = \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{pz^{p-1}} * \frac{g(z)}{z^p} \quad (z \in U).$$

Since

$$\operatorname{Re} \frac{g(z)}{z^p} > \frac{1}{2} \quad (z \in U)$$

and the function

$$\frac{1 + Az}{1 + Bz}$$

is convex (univalent) in U , it follows from (1.8) and Lemma 4 that $(f * g) \in Q_{p,q,s}^n[\alpha_1, A_1, B_1; A, B]$. $\square$

Theorem 7. *Let $\alpha_1 > 0, \nu \in \mathbb{C}^*$ and let $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that*

$$\left| \frac{\frac{\nu\alpha_1}{A_1}(A - B)}{B} - 1 \right| \leq 1$$

or

$$\left| \frac{\frac{\nu\alpha_1}{A_1}(A - B)}{B} + 1 \right| \leq 1$$

if $B \neq 0$, and

$$|\nu\pi| \leq \frac{A_1\pi}{\alpha_1},$$

if $B = 0$.

If $f \in A_n(p)$ with $\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) \neq 0$ for all $z \in U^* = U \setminus \{0\}$, then

$$\frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \prec \frac{1 + Az}{1 + Bz}$$

implies

$$\left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{z^p} \right)^\nu \prec q_1(z),$$

where

$$q_1(z) = \begin{cases} (1 + Bz)^{\frac{\nu\alpha_1}{A_1}(\frac{A-B}{B})}, & \text{if } B \neq 0, \\ e^{\frac{\nu\alpha_1}{A_1}Az}, & \text{if } B = 0, \end{cases}$$

is the best dominant. (All the powers are the principal ones).

Proof. Let us put

$$(3.31) \quad \varphi(z) = \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{z^p} \right)^\nu \quad (z \in U),$$

where the power is the principal one, then $\varphi(z)$ is analytic in U , $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for $z \in U$. Taking the logarithmic derivatives in both sides of (3.31), multiplying by z and using the identity (1.7), we have

$$1 + \frac{z\varphi'(z)}{\frac{\nu\alpha_1}{A_1}\varphi(z)} = \frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \prec \frac{1 + Az}{1 + Bz}.$$

Now the assertions of Theorem 7 follows by using Lemma 5 with $\gamma = \frac{\nu\alpha_1}{A_1}$. This completes the proof of Theorem 7. $\square$

Remark 3. Putting $A_i = 1 (i = 1, \dots, q)$, $B_i = 1 (i = 1, \dots, s)$, $A = 1 - 2\rho$, $0 \leq \rho < 1$ and $B = -1$ in Theorem 7, we obtain the result obtained by Liu [11, Theorem 5].

Putting $A = 1 - \frac{2\eta}{p}$ ($0 \leq \eta < p$), $B = -1$, $A_i = 1 (i = 1, \dots, q)$, $B_i = 1 (i = 1, \dots, s)$, $n = 1$, $q = s + 1$, $\alpha_1 = \beta_1 = p$, $\alpha_j = 1 (j = 2, \dots, s + 1)$, and $\beta_j = 1 (j = 2, \dots, s)$ in Theorem 7, we obtain the following corollary.

Corollary 6. Assume that $\nu \in \mathbb{C}^*$ satisfies either

$$|2\nu(\eta - p) - 1| \leq 1 \text{ or } |2\nu(\eta - p) + 1| \leq 1.$$

If the function $f \in A(p)$ with $f(z) \neq 0$ for $z \in U^*$ satisfy the following inequality:

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \eta (0 \leq \eta < p),$$

then

$$\left(\frac{f(z)}{z^p}\right)^\nu \prec q_2(z) \quad (z \in U),$$

where

$$q_2(z) = (1 - z)^{-2\nu(p-\eta)} (z \in U),$$

is the best dominant.

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