

CONVERGENCE ALMOST EVERYWHERE OF FOURIER SERIES OF THE FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. Let $(\varphi_n(x))$ be an orthonormal system on $[0, 1]$ (ONS). If $f(x)$ is a function of bounded variation ($f(x) \in V$), then the convergence of Fourier series of $f(x)$ generally does not occur (see [2]).

In the present paper we prove that if ONS $(\varphi_n(x))$ satisfies some conditions, then $\lim_{n \rightarrow \infty} S_n(f, x)$ exists a.e. on $[0, 1]$, where $f(x) \in V$ is an arbitrary function.

1. SOME DEFINITIONS AND THEOREMS

Let V be a class of a functions of bounded variation and

$$\|f\|_V = \int_0^1 |f'(x)| dx + \sup_{x \in [0, 1]} |f(x)|$$

be a norm. The sum

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n(f) \varphi_n(x)$$

is the Fourier series of the function $f(x) \in L(0, 1)$.

$$\sigma_n^\alpha(f, x) = \frac{1}{A_n^\alpha} \sum_{k=1}^n A_{n-k}^{\alpha-1} S_k(f; x) \quad (\alpha > 0)$$

is the Césaro mean of the sum $S_n(f; x)$. Let

$$(1.1) \quad P_n(x) = \sum_{k=2^n}^{2^{n+1}-1} c_k \varphi_k(x),$$

where (c_k) are arbitrary numbers, and let

$$(1.2) \quad A(P_n) = \max_{1 \leq i < 2^n} \left| \int_0^{i/2^n} P_n(x) dx \right| = \left| \int_0^{i_n/2^n} P_n(x) dx \right|.$$

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The following theorems are valid (see [1, pp. 87, 132]).

Theorem A. *If $(\varphi_n(x))$ is ONS on $[0, 1]$ and*

$$\sum_{n=1}^{\infty} c_n^2 \log^2 n < +\infty,$$

then

$$(1.3) \quad \sum_{n=1}^{\infty} c_n \varphi_n(x)$$

converges a.e. on $[0, 1]$.

Theorem B. *If $(\varphi_n(x))$ is ONS on $[0, 1]$ and*

$$\sum_{n=1}^{\infty} c_n^2 (\log \log n)^2 < +\infty,$$

then $\lim_{n \rightarrow \infty} \sigma_n^\alpha(f, x)$ exists almost everywhere on $[0, 1]$ for any $\alpha > 0$.

Lemma 1.1. *If $\Phi(x) \in L_2(0, 1)$ and $f(x) \in L_2(0, 1)$ is a function finite in every point of $[0, 1]$, then*

$$(1.4) \quad \int_0^1 f(x) \Phi(x) dx = \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{i/n} \Phi(x) dx \\ + \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(f(x) - f\left(\frac{i}{n}\right) \right) \Phi(x) dx + f(1) \int_0^1 \Phi(x) dx.$$

Proof. It is evident that

$$(1.5) \quad \int_0^1 f(x) \Phi(x) dx = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x) \Phi(x) dx.$$

Applying Abelian transformation we have

$$(1.6) \quad \sum_{i=1}^n f\left(\frac{i}{n}\right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Phi(x) dx = \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{i/n} \Phi(x) dx \\ + f(1) \int_0^1 \Phi(x) dx.$$

Combining (1.5) and (1.6) we get (1.4). □

2. THE MAIN RESULTS

Theorem 2.1. *Let $(\varphi_n(x))$ be ONS on $[0, 1]$ satisfying the following conditions:*

- 1) $\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(1) \log^2 n < +\infty$ $\left(\widehat{\varphi}_n(1) = \int_0^1 \varphi_n(x) dx \right)$;
- 2) *for every sequence of polynomials $(P_n(x))$ (see (1.1) and (1.2))*

$$\sum_{n=1}^{\infty} \left(n \frac{A(P_n)}{\|P_n\|_{L_2}} \right)^2 < +\infty.$$

(Here and below if $A(P_n) = \|P_n\|_{L_2} = 0$, then we assume that $\frac{A(P_n)}{\|P_n\|_{L_2}} = 0$.)

Then for any function $f(x) \in V$

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f) \log^2 n < +\infty.$$

Proof. Suppose that

$$P_n(x) = \sum_{k=2^n}^{2^{n+1}-1} (\widehat{\varphi}_k(f) \log^2 k) \varphi_k(x),$$

then

$$\begin{aligned} (2.1) \quad \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k &= \int_0^1 f(x) \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k(f) \log^2 k \varphi_k(x) dx \\ &= \int_0^1 f(x) P_n(x) dx. \end{aligned}$$

Considering equality (1.4) with $f(x) \in V$ and $\Phi(x) = P_n(x)$, we have

$$\begin{aligned} (2.2) \quad \int_0^1 f(x) P_n(x) dx &= \sum_{i=1}^{2^n-1} \left(f\left(\frac{i}{2^n}\right) - f\left(\frac{i+1}{2^n}\right) \right) \int_0^{\frac{i}{2^n}} P_n(x) dx \\ &\quad + \sum_{i=1}^{2^n} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left(f(x) - f\left(\frac{i}{2^n}\right) \right) P_n(x) dx + f(1) \int_0^1 P_n(x) dx. \end{aligned}$$

If $f(x) \in V$, then

$$\begin{aligned} (2.3) \quad &\left| \sum_{i=1}^{2^n-1} \left(f\left(\frac{i}{2^n}\right) - f\left(\frac{i+1}{2^n}\right) \right) \int_0^{\frac{i}{2^n}} P_n(x) dx \right| \leq \\ &\leq \max_{1 \leq i < 2^n} \left| \int_0^{\frac{i}{2^n}} P_n(x) dx \right| \sum_{i=1}^{2^n-1} \left| f\left(\frac{i}{2^n}\right) - f\left(\frac{i+1}{2^n}\right) \right| < \|f\|_V \cdot A(P_n) \end{aligned}$$

$$\begin{aligned}
&= \|f\|_V \cdot \frac{A(P_n)}{\|P_n\|_{L_2}} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^4 k \right)^{\frac{1}{2}} \\
&\leq 2\|f\|_V \frac{nA(P_n)}{\|P_n\|_{L_2}} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \right)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(2.4) \quad &\left| \sum_{i=1}^{2^n} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left(f(x) - f\left(\frac{i}{2^n}\right) \right) P_n(x) dx \right| \leq \\
&\leq \sum_{i=1}^{2^n} \sup_{x \in [\frac{i-1}{2^n}, \frac{i}{2^n}]} \left| f(x) - f\left(\frac{i}{2^n}\right) \right| \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} |P_n(x)| dx \\
&\leq \left(\int_0^1 P_n^2(x) dx \right)^{\frac{1}{2}} \sum_{i=1}^{2^n} V_{(i-1)2^{-n}}^{2i-n}(f) \cdot 2^{-\frac{n}{2}} \\
&\leq \|f\|_V \cdot 2^{-\frac{n}{2}} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^4 k \right)^{\frac{1}{2}} \\
&\leq 2\|f\|_V n 2^{-\frac{n}{2}} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_0^1 P_n(x) dx \right| &= \left| \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k(f) \log^2 k \int_0^1 \varphi_k(x) dx \right| \\
&\leq \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \right)^{\frac{1}{2}} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) \log^2 k \right)^{\frac{1}{2}}.
\end{aligned}$$

Using (2.3) and (2.4) from (2.2) we get

$$\begin{aligned}
(2.5) \quad &\left| \int_0^1 f(x) P_n(x) dx \right| \leq \\
&\leq 2n\|f\|_V \left(\frac{A(P_n)}{\|P_n\|_{L_2}} + 2^{-\frac{n}{2}} + \frac{1}{n} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) \log^2 k \right)^{\frac{1}{2}} \right) \times \\
&\quad \times \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \right)^{\frac{1}{2}}.
\end{aligned}$$

In such a way (2.1) implies

$$\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \leq 2\|f\|_V n \left(\frac{A(P_n)}{\|P_n\|_{L_2}} + 2^{-\frac{n}{2}} + \frac{1}{n} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) \log^2 k \right)^{\frac{1}{2}} \right) \times$$

$$\times \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \right)^{\frac{1}{2}}.$$

Hence,

$$\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \leq 4 \|f\|_V^2 n^2 \left(\frac{A(P_n)}{\|P_n\|_{L_2}} + 2^{-\frac{n}{2}} + \frac{1}{n} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) \log^2 k \right)^{\frac{1}{2}} \right)^2.$$

Since

$$\sum_{n=1}^{\infty} n^2 2^{-n} < +\infty, \quad \sum_{n=1}^{\infty} \left(\frac{n A(P_n)}{\|P_n\|_{L_2}} \right)^2 < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \widehat{\varphi}_n^2(1) \log^2 n < +\infty,$$

we have

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f) \log^2 n < +\infty.$$

Theorem 2.1 is proved. $\square$

Theorem 2.2. *Let $(\varphi_n(x))$ be ONS on $[0, 1]$ satisfying the conditions of Theorem 2.1, then the Fourier series of any function $f(x) \in A$ converges almost everywhere on $[0, 1]$.*

The validity of Theorem 2.2 follows directly from Theorems A and 2.1.

Theorem 2.3. *Let $(\varphi_n(x))$ be ONS on $[0, 1]$ satisfying conditions:*

- 1) $\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(1) (\log \log n)^2 < +\infty,$
- 2) *for every sequence of polynomials $(P_n(x))$ (see (1.1) and (1.2))*

$$\sum_{n=1}^{\infty} \left(\frac{A(P_n)}{\|P_n\|_{L_2}} \log n \right)^2 < +\infty.$$

Then for any $f(x) \in V$

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f) (\log \log n)^2 < +\infty.$$

Proof. Suppose that

$$P_n(x) = \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k(f) (\log \log k)^2 \varphi_k(x).$$

Analogously to Theorem 2.1 we have (see (2.5))

$$\begin{aligned} \left| \int_0^1 f(x) P_n(x) dx \right| &\leq \\ &\leq 2 \|f\|_V \log n \left(\frac{A(P_n)}{\|P_n\|_{L_2}} + 2^{-\frac{n}{2}} + \frac{1}{\log n} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) (\log \log k)^2 \right)^{\frac{1}{2}} \right) \times \\ &\quad \times \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) (\log \log k)^2 \right)^{\frac{1}{2}}, \end{aligned}$$

According to the equality

$$\begin{aligned} \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) (\log \log k)^2 &= \int_0^1 f(x) \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k(f) (\log \log k)^2 \varphi_k(x) dx \\ &= \int_0^1 f(x) P_n(x) dx \end{aligned}$$

we conclude that

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f) (\log \log n)^2 < +\infty.$$

Theorem 2.3 is proved. $\square$

Theorem 2.4. *Let $(\varphi_n(x))$ be ONS on $[0, 1]$ satisfying conditions of Theorem 2.3. Then for any $f(x) \in A$ and $\alpha > 0$*

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha(f, x)$$

exists a.e. on $[0, 1]$.

The validity of Theorem 2.4 follows from Theorems 2.3 and Theorem B.

Theorem 2.5. *Suppose that for some ONS $(\varphi_n(x))$ the following conditions are satisfied:*

- 1) $\int_0^1 \varphi_n(x) dx = 0$, $n = 1, 2, \dots$;
- 2) *there exist a sequence of polynomials $(P_n(x))$ (see (1.1) and (1.2)) such that*

$$\overline{\lim}_{n \rightarrow \infty} \frac{n A(P_n)}{\|P_n\|_{L_2}} = +\infty.$$

Then there exists a function $f_0(x) \in V$ such that

$$\sum_{n=1}^{\infty} \widehat{\varphi}_k^2(f_0) \log^2 n = +\infty.$$

Proof. Consider the sequence of linear functionals

$$(2.6) \quad C_n(f) = \frac{n}{\|P_n\|_{L_2}} \int_0^1 f(x) P_n(x) dx,$$

where $P_n(x) = \sum_{l=2^n}^{2^{n+1}-1} c_k \varphi_k(x)$ and $(c_k) \in \ell_2$ satisfies condition 2) of Theorem 2.5.

Let

$$(2.7) \quad f_n(x) = \begin{cases} 0 & \text{for } x \in [0, \frac{i_n}{2^n}], \\ 1 & \text{for } x \in [\frac{i_n+1}{2^n}, 1], \\ \text{linear and continuous on} & [\frac{i_n}{2^n}, \frac{i_n+1}{2^n}] \end{cases}$$

be the sequence of functions from V . Obviously,

$$\|f_n\|_V = 2, \quad n = 1, 2, \dots$$

Now, consider (2.2) with $f(x) = f_n(x)$. One can find

$$(2.8) \quad \int_0^1 f_n(x) P_n(x) dx = - \int_0^{\frac{i_n}{2^n}} P_n(x) dx + \int_{\frac{i_n}{2^n}}^{\frac{i_n+1}{2^n}} \left(f_n(x) - f_n\left(\frac{i_n+1}{2^n}\right) \right) P_n(x) dx.$$

Since (see (2.7))

$$(2.9) \quad \left| \int_{\frac{i_n}{2^n}}^{\frac{i_n+1}{2^n}} \left(f_n(x) - f_n\left(\frac{i_n+1}{2^n}\right) \right) P_n(x) dx \right| \leq 2^{-\frac{n}{2}} \left(\int_0^1 P_n^2(x) dx \right)^{\frac{1}{2}} = 2^{-\frac{n}{2}} \|P_n\|_{L_2},$$

from (2.8) we get

$$\left| \int_0^1 f_n(x) P_n(x) dx \right| \geq A(P_n) - 2^{-\frac{n}{6}} \|P_n\|_{L_2}.$$

Taking into account condition 2) of Theorem 2.5 we obtain

$$\frac{n}{\|P_n\|_{L_2}} \left| \int_0^1 f_n(x) P_n(x) dx \right| \geq \frac{nA(P_n)}{\|P_n\|_{L_2}} A(P_n) - n2^{-\frac{n}{2}}.$$

Consequently (see (2.6)),

$$(2.10) \quad \overline{\lim}_{n \rightarrow \infty} |C_n(f_n)| = +\infty.$$

From (2.10) and Banach-Steinhaus theorem it follows that there exists a function $f_0(x) \in A$ such that

$$(2.11) \quad \overline{\lim}_{n \rightarrow \infty} |C_n(f_0)| = +\infty.$$

Next, by virtue of Hölder's inequality we get

$$\begin{aligned} \left| \int_0^1 f_0(x) P_n(x) dx \right| &= \left| \sum_{k=2^n}^{2^{n+1}-1} c_k \int_0^1 f_0(x) \varphi_k(x) dx \right| \\ &= \left| \sum_{k=2^n}^{2^{n+1}-1} c_k \widehat{\varphi}_k(f_0) \right| \leq \left(\sum_{k=2^n}^{2^{n+1}-1} c_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0) \right)^{\frac{1}{2}} \\ &\leq \frac{\|P_n\|_{L_2}}{n} \left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0) \log^2 k \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\left(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0) \log^2 k \right)^{\frac{1}{2}} \geq \frac{n}{\|P_n\|_{L_2}} \left| \int_0^1 f_0(x) P_n(x) dx \right| = C_n(f_0).$$

Then from (2.11) we have

$$\overline{\lim}_{n \rightarrow \infty} \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0) \log^2 k = +\infty.$$

Theorem 2.5 is completely proved. $\square$

Theorem 2.6. *If $(\varphi_n(x))$ is ONS on $[0, 1]$ satisfying conditions 1) and 2) of Theorem 2.5, then there exists a function $f_0(x) \in A$ such that*

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f_0) (\log \log n)^2 = +\infty.$$

Theorem 2.6 is proved analogously to Theorem 2.5.

Remark 2.1. If in Theorem 2.5 we change condition $\int_0^1 \varphi_n(x) dx = 0$ by the inequality $\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(1) \log^2 n < +\infty$, then Theorem 2.5 will also be valid.

3. EFFICIENCY OF THEOREM 2.1

Consider trigonometric system $(\sqrt{2} \cos 2\pi nx, \sqrt{2} \sin 2\pi nx)$ on $[0, 1]$ for every sequences (a_k) and (b_k) . Then

$$\begin{aligned} A(P_n) &= \left| \int_0^{\frac{1}{2^n}} \sum_{k=2^n}^{2^{n+1}-1} \sqrt{2} (a_k \sin 2\pi kx + b_k \cos 2\pi kx) dx \right| \\ &\leq 2 \sum_{k=2^n}^{2^{n+1}-1} \frac{|a_k| + |b_k|}{k} \leq 4 \left(\sum_{k=2^n}^{2^{n+1}-1} (a_k^2 + b_k^2) \right)^{\frac{1}{2}} \left(\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k^2} \right)^{\frac{1}{2}} \\ &\leq 2^{-\frac{n}{2}+2} \|P_n\|_{L_2}. \end{aligned}$$

Consequently, for the trigonometric system conditions of Theorem 2.1 are valid.

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