

THE MAXIMAL OPERATORS OF LOGARITHMIC MEANS OF ONE-DIMENSIONAL VILENKIN-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to investigate (H_p, L_p) -type inequalities for maximal operators of logarithmic means of one-dimensional bounded Vilenkin-Fourier series.

1. INTRODUCTION

In one-dimensional case the weak type inequality

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1 \quad (\lambda > 0)$$

can be found in Zygmund [20] for the trigonometric series, in Schipp [11] for Walsh series and in Pál, Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujji [3] and Simon [12] verified that σ^* is bounded from H_1 to L_1 . Weisz [17] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the space L_p for $p > 1/2$. Simon [13] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava ([7], see also [2]).

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. We mention, for instance, the paper by Szász [14] and Yabuta [19]. This means with respect to the Walsh and Vilenkin systems was discussed by Simon[13] and Gát[4].

Móricz and Siddiqi[9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p function in norm. The case when $q_k = 1/k$ is excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [5] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1 . Among there, they gave a negative answer to the question of Móricz and

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Siddiqi [9]. Gát and Goginava [6] proved that for each measurable function $\phi(u) = o(u\sqrt{\log u})$ there exists an integrable function f , such that

$$\int_{G_m} \phi(|f(x)|) d\mu(x) < \infty$$

and there exist a set with positive measure, such that the Walsh-logarithmic means of the function diverge on this set.

The main aim of this paper is to investigate (H_p, L_p) -type inequalities for the maximal operators of Riesz and Nörlund logarithmic means of one-dimensional Vilenkin-Fourier series. We prove that the maximal operator R^* is bounded from the Hardy space H_p to the space L_p when $p > 1/2$. We also shows that when $0 < p \leq 1/2$ there exists a martingale $f \in H_p$ for which

$$\|R^*f\|_{L_p} = +\infty.$$

For the Nörlund logarithmic means we prove that when $0 < p \leq 1$ there exists a martingale $f \in H_p$ for which

$$\|L^*f\|_{L_p} = +\infty.$$

Analogical theorems for Walsh-Paley system is proved in [8].

2. DEFINITIONS AND NOTATION

Let N_+ denote the set of the positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ the addition group of integers modulo m_k .

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of the groups Z_{m_j} .

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_{m_k} with $\mu(G_m) = 1$.

If the sequence m is bounded then G_m is called a bounded Vilenkin group, else it is called an unbounded one. *In this paper we discuss bounded Vilenkin groups only.* The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \quad (x_i \in Z_{m_j}).$$

It is easy to give a base for the neighborhood of G_m

$$I_0(x) := G_m$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in N)$$

Denote $I_n := I_n(0)$ for $n \in N_+$.

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in N),$$

then every $n \in N$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$ where $n_j \in Z_{m_j}$ ($j \in N_+$) and only a finite number of n_j s differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first, define the complex valued function $r_k(x) : G_m \rightarrow C$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad (i^2 = -1, x \in G_m, k \in N).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in N).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 15].

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\begin{aligned} \widehat{f}(k) &:= \int_{G_m} f \bar{\psi}_k d\mu \quad (k \in N), \\ S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \quad (n \in N_+, S_0 f := 0), \\ \sigma_n f &:= \frac{1}{n} \sum_{k=0}^{n-1} S_k f \quad (n \in N_+), \\ D_n &:= \sum_{k=0}^{n-1} \psi_k \quad (n \in N_+). \end{aligned}$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n. \end{cases}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \quad (0 < p < \infty).$$

The σ -algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ is denoted by $F_n (n \in N)$. Denote by $f = (f^{(n)}, n \in N)$ a martingale with respect to $F_n (n \in N)$ (for details see e.g. [16]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in N} |f^{(n)}|.$$

In case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in N} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingale for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in N)$ is a martingale.

If $f = (f^{(n)}, n \in N)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\hat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f^{(k)}(x) \bar{\Psi}_i(x) d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in N)$ obtained from f .

In the literature, there is the notion of Riesz's logarithmic means of the Fourier series. The n -th Riesz's logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n f(x) := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k f(x)}{k},$$

where

$$l_n := \sum_{k=1}^n (1/k).$$

Let $\{q_k : k > 0\}$ be a sequence of nonnegative numbers. The n -th Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n} \sum_{k=1}^n q_{n-k} S_k f,$$

where

$$Q_n := \sum_{k=1}^n q_k.$$

If $q_k = 1/k$, then we get Nörlund logarithmic means

$$L_n f(x) := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k f(x)}{n-k}$$

It is a kind of "reverse" Riesz's logarithmic mean. In this paper we call these means logarithmic means.

For the martingale f we consider the following maximal operators of

$$\begin{aligned} R^*f(x) &:= \sup_{n \in N} |R_n f(x)|, \\ L^*f(x) &:= \sup_{n \in N} |L_n f(x)|, \\ \sigma^*f(x) &:= \sup_{n \in N} |\sigma_n f(x)|. \end{aligned}$$

A bounded measurable function a is a p -atom, if there exists a dyadic interval I , such that

- a) $\int_I a d\mu = 0$,
- b) $\|a\|_\infty \leq \mu(I)^{-1/p}$,
- c) $\text{supp}(a) \subset I$.

3. FORMULATION OF MAIN RESULTS

Theorem 1. *Let $p > 1/2$. Then the maximal operator R^* is bounded from the Hardy space H_p to the space L_p .*

Theorem 2. *Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that*

$$\|R^*f\|_p = +\infty.$$

Corollary 1. *Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that*

$$\|\sigma^*f\|_p = +\infty.$$

Theorem 3. *Let $0 < p \leq 1$. Then there exists a martingale $f \in L_p$ such that*

$$\|L^*f\|_p = +\infty.$$

4. AUXILIARY PROPOSITIONS

Lemma 1. [18] *A martingale $f = (f^{(n)}, n \in N)$ is in H_p ($0 < p \leq 1$) if and only if there exists a sequence $(a_k, k \in N)$ of p -atoms and a sequence $(\mu_k, k \in N)$ of a real numbers such that for every $n \in N$*

$$(1) \quad \begin{aligned} \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k &= f^{(n)}, \\ \sum_{k=0}^{\infty} |\mu_k|^p &< \infty. \end{aligned}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{K=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (1).

5. PROOF OF THE THEOREMS

Proof of Theorem 1: Using Abel transformation we obtain

$$R_n f(x) = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\sigma_j f(x)}{j+1} + \frac{\sigma_n f(x)}{l_n},$$

Consequently,

$$(2) \quad L^* f \leq c \sigma^* f.$$

On the other hand Weisz[17] proved that σ^* is bounded from the Hardy space H_p to the space L_p when $p > 1/2$. Hence, from (2) we conclude that R^* is bounded from the martingale Hardy space H_p to the space L_p when $p > 1/2$. $\square$

Proof of Theorem 2: Let $\{\alpha_k : k \in N\}$ be an increasing sequence of the positive integers such that

$$(3) \quad \sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty,$$

$$(4) \quad \sum_{\eta=0}^{k-1} \frac{(M_{2\alpha_\eta})^{1/p}}{\sqrt{\alpha_\eta}} < \frac{(M_{2\alpha_k})^{1/p}}{\sqrt{\alpha_k}},$$

$$(5) \quad \frac{(M_{2\alpha_{k-1}})^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{M_{\alpha_k}}{\alpha_k^{3/2}}.$$

We note that such an increasing sequence $\{\alpha_k : k \in N\}$ which satisfies conditions (3)-(5) can be constructed.

Let

$$f^{(A)}(x) = \sum_{\{k; 2\alpha_k < A\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{m_{2\alpha_k}}{\sqrt{\alpha_k}}$$

and

$$a_k(x) = \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} \left(D_{M(2\alpha_k+1)}(x) - D_{M_{2\alpha_k}}(x) \right).$$

It is easy to show that

$$\|a_k\|_\infty \leq \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} M_{2\alpha_k+1} \leq (M_{2\alpha_k})^{1/p} = (\mu(\text{supp } a_k))^{-1/p},$$

$$(6) \quad S_{M_A} a_k(x) = \begin{cases} a_k(x), & 2\alpha_k < A, \\ 0, & 2\alpha_k \geq A. \end{cases}$$

$$\begin{aligned}
f^{(A)}(x) &= \sum_{\{k; 2\alpha_k < A\}} \lambda_k a_k = \sum_{k=0}^{\infty} \lambda_k S_{M_A} a_k(x), \\
\text{supp}(a_k) &= I_{2\alpha_k}, \\
\int_{I_{2\alpha_k}} a_k d\mu &= 0.
\end{aligned}$$

From (3) and Lemma 1 we conclude that $f = (f^{(n)}, n \in N) \in H_p$.
Let

$$q_A^s = M_{2A} + M_{2s} - 1, \quad A > S.$$

Then we can write

$$\begin{aligned}
(7) \quad R_{q_{\alpha_k}^s} f(x) &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{q_{\alpha_k}^s} \frac{S_j f(x)}{j} \\
&= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{S_j f(x)}{j} + \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{S_j f(x)}{j} = I + II.
\end{aligned}$$

It is easy to show that

$$(8) \quad \widehat{f}(j) = \begin{cases} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}, & \text{if } j \in \{M_{2\alpha_k}, \dots, M_{2\alpha_{k+1}} - 1\}, k = 0, 1, 2, \dots, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{M_{2\alpha_k}, \dots, M_{2\alpha_{k+1}} - 1\}. \end{cases}$$

Let $j < M_{2\alpha_k}$. Then from (4) and (8) we have

$$\begin{aligned}
(9) \quad |S_j f(x)| &\leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_{\eta+1}}-1} |\widehat{f}(v)| \\
&\leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_{\eta+1}}-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}} \leq \frac{c M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.
\end{aligned}$$

Consequently,

$$(10) \quad |I| \leq \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{|S_j f(x)|}{j} \leq \frac{c}{\alpha_k} \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{1}{j} \leq c \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.$$

Let $M_{2\alpha_k} \leq j \leq q_{\alpha_k}^s$. Then we have the following

$$\begin{aligned}
 (11) \quad S_j f(x) &= \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_{\eta+1}}-1} \widehat{f}(v) \psi_v(x) + \sum_{v=M_{2\alpha_k}}^{j-1} \widehat{f}(v) \psi_v(x) \\
 &= \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_{\eta+1}}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \\
 &\quad + \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left(D_j(x) - D_{M_{2\alpha_k}}(x) \right).
 \end{aligned}$$

This gives that

$$\begin{aligned}
 (12) \quad II &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{1}{j} \left(\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_{\eta+1}}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \right) \\
 &\quad + \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{\left(D_j(x) - D_{M_{2\alpha_k}}(x) \right)}{j} = II_1 + II_2.
 \end{aligned}$$

To discuss II_1 , we use (4). Thus, we can write that

$$(13) \quad |II_1| \leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}} \leq \frac{cM_{2\alpha_{k-1}}}{\sqrt{\alpha_{k-1}}}.$$

Since,

$$(14) \quad D_{j+M_{2\alpha_k}}(x) = D_{M_{2\alpha_k}}(x) + \psi_{M_{2\alpha_k}}(x) D_j(x), \quad \text{when } j < M_{2\alpha_k},$$

for II_2 we have,

$$\begin{aligned}
 (15) \quad II_2 &= \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=0}^{M_{2s}} \frac{D_{j+M_{2\alpha_k}}(x) - D_{M_{2\alpha_k}}(x)}{j+M_{2\alpha_k}} \\
 &= \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \psi_{M_{2\alpha_k}} \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}}.
 \end{aligned}$$

We write

$$R_{q_{\alpha_k}^s} f(x) = I + II_1 + II_2,$$

Then by (5), (7), (10) and (12)-(15) we have

$$\begin{aligned}
 \left| R_{q_{\alpha_k}^s} f(x) \right| &\geq |II_2| - |I| - |II_1| \geq |II_2| - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}} \\
 &\geq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left| \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}} \right| - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}}
 \end{aligned}$$

Let $0 < p \leq 1/2$, $x \in I_{2s} \setminus I_{2s+1}$ for $s = [\alpha_k/2], \dots, \alpha_k$. Then it is evident

$$\left| \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}} \right| \geq \frac{cM_{2s}^2}{M_{2\alpha_k}}$$

Hence, we can write

$$\begin{aligned} \left| R_{q_{\alpha_k}^s} f(x) \right| &\geq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \frac{cM_{2s}^2}{M_{2\alpha_k}} - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}} \\ &\geq \frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}} - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}} \geq \frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}}. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{G_m} |R^* f(x)|^p d\mu(x) &\geq \sum_{s=[\alpha_k/2]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left| R_{q_{\alpha_k}^s} f(x) \right|^p d\mu(x) \\ &\geq \sum_{s=[\alpha_k/2]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left(\frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}} \right)^p d\mu(x) \\ &\geq c \sum_{s=[\alpha_k/2]}^{\alpha_k} \frac{M_{2\alpha_k}^{1-2p} M_{2s}^{2p-1}}{\alpha_k^{3p/2}} \\ &\geq \begin{cases} \frac{2^{\alpha_k(1-2p)}}{\alpha_k^{3p/2}}, & \text{when } 0 < p < 1/2 \\ c\alpha_k^{1/4}, & \text{when } p = 1/2 \end{cases} \rightarrow \infty, \text{ when } k \rightarrow \infty. \end{aligned}$$

which completes the proof of the Theorem 2. $\square$

Proof of Theorem 3: We write

$$\begin{aligned} (16) \quad L_{q_{\alpha_k}^s} f(x) &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{q_{\alpha_k}^s} \frac{S_j f(x)}{q_{\alpha_k}^s - j} \\ &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{S_j f(x)}{q_{\alpha_k}^s - j} + \frac{1}{q_{\alpha_k}^s} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{S_j f(x)}{q_{\alpha_k}^s - j} = III + IV. \end{aligned}$$

Since (see 9)

$$|S_j f(x)| \leq c \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}, \quad j < M_{2\alpha_k}.$$

For *III* we can write

$$(17) \quad |III| \leq \frac{c}{\alpha_k} \sum_{j=0}^{M_{2\alpha_k}-1} \frac{1}{q_{\alpha_k}^s - j} \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}} \leq c \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.$$

Using (11) we have

$$(18) IV = \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{1}{q_{\alpha_k, s} - j} \left(\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \right) \\ + \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{\left(D_j(x) - D_{M_{2\alpha_k}}(x) \right)}{q_{\alpha_k}^s - j} = IV_1 + IV_2.$$

Applying (4) in IV_1 we have

$$(19) \quad |IV_1| \leq c \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}},$$

From (14) we obtain

$$(20) \quad IV_2 = \frac{1}{l_{q_{\alpha_k, s}}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \psi_{M_{2\alpha_k}} \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{M_{2s} - j}.$$

Let $x \in I_{2s} \setminus I_{2s+1}$. Then $D_j(x) = j$, $j < M_{2s}$. Consequently,

$$\sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{M_{2s} - j} = \sum_{j=0}^{M_{2s}-1} \frac{j}{M_{2s} - j} = \sum_{j=0}^{M_{2s}-1} \left(\frac{M_{2s}}{M_{2s} - j} - 1 \right) \geq csM_{2s}.$$

Then

$$(21) \quad |IV_2| \geq c \frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s}, \quad x \in I_{2s} \setminus I_{2s+1}.$$

Combining (5), (16)-(21) for $x \in I_{2s} \setminus I_{2s+1}$, $s = [\alpha_k/2], \dots, \alpha_k$, and $0 < p \leq 1$ we have

$$\left| L_{q_{\alpha_k}^s} f(x) \right| \geq c \frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s} - c \frac{M_{\alpha_k}}{\alpha_k} \geq c \frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s}$$

Then

$$\begin{aligned}
 \int_{G_m} |L^* f(x)|^p d\mu(x) &\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} |L^* f(x)|^p d\mu(x) \\
 &\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} \left| L_{q_{\alpha_k}^s} f(x) \right|^p d\mu(x) \\
 &\geq c \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} \left(\frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s} \right)^p d\mu(x) \\
 &\geq c \sum_{s=[m_k/2]}^{m_k} \frac{M_{2\alpha_{k-1}}^{1-p}}{\alpha_k^{p/2}} M_{2s}^{p-1} \\
 &\geq \begin{cases} \frac{2^{\alpha_k(1-p)}}{\alpha_k^{p/2}}, & \text{when } 0 < p < 1, \\ c\sqrt{\alpha_k}, & \text{when } p = 1, \end{cases} \rightarrow \infty, \text{ when } k \rightarrow \infty.
 \end{aligned}$$

Theorem 3 is proved. $\square$

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