

ON THE CONVERGENCE ALMOST EVERYWHERE OF DOUBLE SERIES WITH RESPECT TO DIAGONAL BLOCK-ORTHONORMAL SYSTEMS

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ABSTRACT. The diagonal double block-orthonormal system is introduced. The two-dimensional generalization of Menshov-Rademacher's and V.F. Gaposhkin's theorems on the almost everywhere convergence of series with respect to block-orthonormal systems is proved.

Block-orthonormal systems were introduced by Gaposhkin [2]. He proved, that the Menshov-Rademacher's theorem [1] and the strong law of large numbers are valid for such systems in certain conditions. In [3] were obtained some results on convergence and summability of series with respect to block-orthonormal systems. In particular, Menshov-Rademacher's and Gaposhkin's theorems were generalized and the exact Weyl multipliers for the convergence and summability almost everywhere of series with respect to block-orthogonal systems were established in the cases, when Menshov-Rademacher's and Gaposhkin's theorems are not true.

The two-dimensional analog of Menshov-Rademacher's theorem was obtained in [4]. In [5] was considered the almost everywhere convergence of multiple orthogonal series.

In the present paper it will be introduced a diagonal block-orthonormal systems and it will be considered the almost everywhere convergence of double series with respect to diagonal block-orthonormal systems.

Definition 1. Let $\{M_k\}$ and $\{N_k\}$ be the increasing sequences of natural numbers and $\Delta_k = ([1, M_{k+1}] \times [1, N_{k+1}]) \setminus ([1, M_k] \times [1, N_k])$, $(k \geq 1)$. Let $\{\varphi_{mn}\}$ be a system of functions from $L^2((0, 1)^2)$. The system $\{\varphi_{mn}\}$ will be called a *diagonal Δ_k -orthonormal system (Δ_k -ONS)* if:

1. $\|\varphi_{mn}\|_2 = 1$, $m = 1, 2, \dots, n = 1, 2, \dots$;
2. $(\varphi_{ij}, \varphi_{pq}) = 0$, for $(i, j), (p, q) \in \Delta_k$, $(i, j) \neq (p, q)$, $(k \geq 1)$.

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Let the sequences $\{M_k\}$, $\{N_k\}$ be fixed and $\{\varphi_{mn}\}$ be a diagonal Δ_k -ONS. Let the double series

$$(1) \quad \sum_{m,n=1}^{\infty} a_{mn} \varphi_{mn}(x, y)$$

Is given, where $\sum_{m,n=1}^{\infty} a_{mn}^2 < \infty$.

Under the convergence of the series (1) it is understood the convergence in Pringsame's sense, that is the existence of the limit

$$(2) \quad \lim_{M,N \rightarrow \infty} \sum_{m=1}^M \sum_{n=1}^N a_{mn} \varphi_{mn}(x, y),$$

as M and N independently approaches infinity.

Definition 2. Let $\{\omega(m, n)\}$ be a sequence of positive numbers, for which $\omega(m, n) \leq \omega(m, n+1)$ and $\omega(m, n) \leq \omega(m+1, n)$ ($m, n = 1, 2, \dots$). The sequence $\{\omega(m, n)\}$ will be called the *Weyl multiplier* for the convergence almost everywhere (a. e.) of series (1) with respect to diagonal Δ_k -ONS $\{\varphi_{mn}\}$ if the convergence of the series $\sum_{m,n=1}^{\infty} a_{mn}^2 \omega(m, n) < \infty$ guarantees the existence of the limit (2) a. e. on $(0, 1)^2$.

In this paper, the logarithms are to the base 2.

Theorem. Let the sequences $\{M_k\}$, $\{N_k\}$ be fixed and $\{\omega_1(m)\}$, $\{\omega_2(n)\}$ be the nondecreasing sequences of positive numbers. In order that a double sequence $\{\omega_1(m)\omega_2(n)\}$ be the Weyl multiplier for the convergence a. e. of series (1) with respect to all diagonal Δ_k -ONS $\{\varphi_{mn}\}$, it is necessary and sufficient that the following two conditions be fulfilled:

$$(3) \quad \sum_{p,q=1}^{\infty} \frac{1}{\omega_1(M_p)\omega_2(N_q)} < \infty,$$

$$(4) \quad \log^2 m = O(\omega_i(m)), \quad i = 1, 2, \quad (m \rightarrow \infty).$$

Below we shall use the following lemma, which is the two-dimensional analog of well-known lemma: (see [1, Lemma 2.3.1], [4, lemma 1]).

Lemma 1. Let $\{\varphi_{mn}\}$ be a orthonormal system from $L^2((0, 1)^2)$. Then for all numbers $\{a_{mn}\}_{0 \leq m \leq M, 0 \leq n \leq N}$ are fulfilled:

$$(5) \quad \int_0^1 \int_0^1 \max_{\substack{0 \leq m \leq M \\ 0 \leq n \leq N}} \left| \sum_{i=0}^m \sum_{j=0}^n a_{ij} \varphi_{ij}(x, y) \right|^2 dx dy \\ \leq c \log^2(M+2) \log^2(N+2) \sum_{i=0}^M \sum_{j=0}^N a_{ij}^2$$

$$(6) \quad \int_0^1 \int_0^1 \max_{0 \leq m \leq M} \left| \sum_{i=0}^m \sum_{j=0}^N a_{ij} \varphi_{ij}(x, y) \right|^2 dx dy \leq c \log^2(M+2) \sum_{i=0}^M \sum_{j=0}^N a_{ij}^2,$$

$$(7) \quad \int_0^1 \int_0^1 \max_{0 \leq n \leq N} \left| \sum_{i=0}^M \sum_{j=0}^n a_{ij} \varphi_{ij}(x, y) \right|^2 dx dy \leq c \log^2(N+2) \sum_{i=0}^M \sum_{j=0}^N a_{ij}^2.$$

For (5), (6) and (7) we have generalizations of Kantorovich ([1, p. 89]):

$$(8) \quad \int_0^1 \int_0^1 \max_{\substack{0 \leq m \leq M, \\ 0 \leq n \leq N}} \left| \sum_{i=0}^m \sum_{j=0}^n a_{ij} \varphi_{ij}(x, y) \right|^2 dx dy \\ \leq c \sum_{i=0}^M \sum_{j=0}^N a_{ij}^2 \log^2(i+2) \log^2(j+2),$$

$$(9) \quad \int_0^1 \int_0^1 \max_{0 \leq m \leq M} \left| \sum_{i=0}^m \sum_{j=0}^N a_{ij} \varphi_{ij}(x, y) \right|^2 dx dy \leq c \sum_{i=0}^M \sum_{j=0}^N a_{ij}^2 \log^2(i+2),$$

$$(10) \quad \int_0^1 \int_0^1 \max_{0 \leq n \leq N} \left| \sum_{i=0}^M \sum_{j=0}^n a_{ij} \varphi_{ij}(x, y) \right|^2 dx dy \leq c \sum_{i=0}^M \sum_{j=0}^N a_{ij}^2 \log^2(i+2).$$

Proof of Theorem. Sufficiency. Let for sequence $\{\omega_1(m)\omega_2(n)\}$ the conditions (3), (4) are fulfilled and let for sequence $\{a_{mn}\}$ have:

$$\sum_{m,n=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) < \infty.$$

Let $\{\varphi_{mn}\}$ be arbitrary diagonal Δ_k -ONS. In first we shall prove that the limit

$$(11) \quad \lim_{p,q \rightarrow \infty} S_{M_p, N_q}(x, y) = \sum_{m=1}^{M_p} \sum_{n=1}^{N_q} a_{mn} \varphi_{mn}(x, y)$$

exists almost everywhere on $(0, 1)^2$.

Without loss of generality it can be assumed that $M_0 = N_0 = 0$ and $\omega_1(0) = \omega_2(0) = 1$. Then

$$\begin{aligned} & |S_{M_{p+s}, N_{q+r}}(x, y) - S_{M_p, N_q}(x, y)| \\ &= \left| \sum_{m=M_p+1}^{M_{p+s}} \sum_{n=1}^{N_{q+r}} a_{mn} \varphi_{mn}(x, y) + \sum_{m=1}^{M_p} \sum_{n=N_q+1}^{N_{q+r}} a_{mn} \varphi_{mn}(x, y) \right| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=p}^{\infty} \sum_{j=0}^{\infty} \left| \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right| \\ &+ \sum_{i=0}^{\infty} \sum_{j=q}^{\infty} \left| \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right| = I_p(x, y) + J_q(x, y). \end{aligned}$$

We shall prove that the double series

$$(12) \quad \sum_{i,j=0}^{\infty} \left| \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right|$$

converges a. e. on $(0, 1)^2$. Indeed, we have

$$\begin{aligned} &\sum_{i,j=0}^{\infty} \int_0^1 \int_0^1 \left| \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right| dx dy \\ &\leq \sum_{i,j=0}^{\infty} \left(\int_0^1 \int_0^1 \left| \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right|^2 dx dy \right)^{\frac{1}{2}} \\ &= \sum_{i,j=0}^{\infty} \left(\sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn}^2 \right)^{\frac{1}{2}} \leq c \left(\sum_{i,j=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) \right)^{\frac{1}{2}} < \infty \end{aligned}$$

Hence Levi's theorem implies, that series (12) converges a. e. on $(0, 1)^2$. Then almost everywhere on $(0, 1)^2$ we have

$$\lim_{p \rightarrow \infty} I_p(x, y) = 0 \text{ and } \lim_{q \rightarrow \infty} J_q(x, y) = 0.$$

Therefore the limit (11) exists almost everywhere on $(0, 1)^2$.

Let k, l be the natural numbers, for which

$$M_p < k \leq M_{p+1}, \quad N_q < l \leq N_{q+1}.$$

We have

$$\begin{aligned} &\max_{\substack{M_p < k \leq M_{p+1} \\ N_q < l \leq N_{q+1}}} |S_{k,l}(x, y) - S_{M_p, N_q}(x, y)| \\ &\leq \max_{N_q < l \leq N_{q+1}} \left| \sum_{m=1}^{M_p} \sum_{n=N_q+1}^l a_{mn} \varphi_{mn}(x, y) \right| + \max_{M_p < k \leq M_{p+1}} \left| \sum_{m=M_p+1}^k \sum_{n=1}^{N_q} a_{mn} \varphi_{mn}(x, y) \right| \\ &+ \max_{\substack{M_p < k \leq M_{p+1} \\ N_q < l \leq N_{q+1}}} \left| \sum_{m=M_p+1}^k \sum_{n=N_q+1}^l a_{mn} \varphi_{mn}(x, y) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{\infty} \sup_{N_q \leq l_1 < l_2 < \infty} \left| \sum_{m=M_{i+1}}^{M_{i+1}} \sum_{n=l_1+1}^{l_2} a_{mn} \varphi_{mn}(x, y) \right| \\
&+ \sum_{j=0}^{\infty} \sup_{M_p \leq k_1 < k_2 < \infty} \left| \sum_{m=k_1+1}^{k_2} \sum_{n=N_{j+1}}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right| \\
&+ \max_{\substack{M_p < k \leq M_{p+1} \\ N_q < l \leq N_{q+1}}} \left| \sum_{m=M_p+1}^k \sum_{n=N_q+1}^l a_{mn} \varphi_{mn}(x, y) \right| \\
&= \sum_{i=0}^{\infty} \alpha_i^q(x, y) + \sum_{j=0}^{\infty} \beta_j^p(x, y) + \delta_{p,q}(x, y).
\end{aligned}$$

It's clear, that the sequences

$$\alpha_q(x, y) = \sum_{i=0}^{\infty} \alpha_i^q(x, y) \text{ and } \beta_p(x, y) = \sum_{j=0}^{\infty} \beta_j^p(x, y)$$

are increasing sequences. Show that a. e. on $(0, 1)^2$

$$\lim_{q \rightarrow \infty} \alpha_q(x, y) = 0 \text{ and } \lim_{p \rightarrow \infty} \beta_p(x, y) = 0.$$

Indeed, using lemma we have:

$$\begin{aligned}
&\sum_{i=0}^{\infty} \int_0^1 \int_0^1 \alpha_i^q(x, y) dx dy \leq \sum_{i=0}^{\infty} \left(\int_0^1 \int_0^1 [\alpha_i^q(x, y)]^2 dx dy \right)^{\frac{1}{2}} \\
&\leq \sum_{i=0}^{\infty} \left(\int_0^1 \int_0^1 \sup_{N_q \leq l_1 < l_2 < \infty} \left| \sum_{m=M_{i+1}}^{M_{i+1}} \sum_{n=N_q+1}^{l_2} a_{mn} \varphi_{mn}(x, y) \right. \right. \\
&\quad \left. \left. - \sum_{m=M_{i+1}}^{M_{i+1}} \sum_{n=N_q+1}^{l_1} a_{mn} \varphi_{mn}(x, y) \right|^2 dx dy \right)^{\frac{1}{2}} \\
&\leq c \sum_{i=0}^{\infty} \left(\int_0^1 \int_0^1 \sup_{N_q < l} \left| \sum_{m=M_{i+1}}^{M_{i+1}} \sum_{n=N_q+1}^l a_{mn} \varphi_{mn}(x, y) \right|^2 dx dy \right)^{\frac{1}{2}} \\
&\leq c \sum_{i=0}^{\infty} \left[\int_0^1 \int_0^1 \left(\sum_{j=q}^{\infty} \left| \sum_{m=M_{i+1}}^{M_{i+1}} \sum_{n=N_{j+1}}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right| \right)^2 dx dy \right]^{\frac{1}{2}} \\
&\quad + c \sum_{i=0}^{\infty} \left[\int_0^1 \int_0^1 \sum_{j=q}^{\infty} \max_{N_j < l \leq N_{j+1}} \left| \sum_{m=M_{i+1}}^{M_{i+1}} \sum_{n=N_j+1}^l a_{mn} \varphi_{mn}(x, y) \right|^2 dx dy \right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq c \sum_{i=0}^{\infty} \left[\left(\sum_{j=q}^{\infty} \int_0^1 \int_0^1 \left| \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right|^2 dx dy \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\sum_{j=q}^{\infty} \int_0^1 \int_0^1 \max_{N_j < l \leq N_{j+1}} \left| \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^l a_{mn} \varphi_{mn}(x, y) \right|^2 dx dy \right)^{\frac{1}{2}} \right] \\
&\leq c \sum_{i=0}^{\infty} \left[\sum_{i=q}^{\infty} \left(\sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} \alpha_{mn}^2 \right)^{\frac{1}{2}} + \left(\sum_{i=q}^{\infty} \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} \alpha_{mn}^2 \log^2(n+2) \right)^{\frac{1}{2}} \right] \\
&\leq c \left(\sum_{i=0}^{\infty} \sum_{i=q}^{\infty} \left(\sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} \alpha_{mn}^2 \right) \omega_1(M_i) \omega_2(N_j) \right)^{\frac{1}{2}} \cdot \left(\sum_{i=0}^{\infty} \sum_{i=q}^{\infty} \frac{1}{\omega_1(M_i) \omega_2(N_j)} \right)^{\frac{1}{2}} \\
&\leq c \left(\sum_{i=0}^{\infty} \sum_{i=q}^{\infty} \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} \alpha_{mn}^2 \log^2(n+2) \omega_1(M_i) \right)^{\frac{1}{2}} \cdot \left(\sum_{i=0}^{\infty} \frac{1}{\omega_1(M_i)} \right)^{\frac{1}{2}} \\
&\leq c \left(\sum_{m=1}^{\infty} \sum_{n=N_q+1}^{\infty} \alpha_{mn}^2 \omega_1(m) \omega_2(n) \right)^{\frac{1}{2}},
\end{aligned}$$

hence

$$\lim_{q \rightarrow \infty} \int_0^1 \int_0^1 \alpha_q(x, y) dx dy = 0.$$

Then by Fatou's theorem

$$(13) \quad \lim_{q \rightarrow \infty} \alpha_q(x, y) = 0 \text{ a. e. on } (0, 1)^2.$$

Similarly we obtain

$$(14) \quad \lim_{p \rightarrow \infty} \beta_p(x, y) = 0 \text{ a. e. on } (0, 1)^2.$$

Now we prove that

$$(15) \quad \lim_{p, q \rightarrow \infty} \delta_{p, q}(x, y) = 0 \text{ a. e. on } (0, 1)^2.$$

Indeed, using inequality (8) we get

$$\begin{aligned}
&\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \int_0^1 \int_0^1 \delta_{p, q}^2(x, y) dx dy \\
&\leq c \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=M_p+1}^{M_{p+1}} \sum_{n=N_q+1}^{N_{q+1}} a_{mn}^2 \log^2(m+2) \log^2(n+2)
\end{aligned}$$

$$\leq c \sum_{m,n=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) < \infty,$$

hence $\sum_{p,q=0}^{\infty} \delta_{p,q}^2(x, y) < \infty$ almost everywhere on $(0, 1)^2$. Then we obtain (15). Therefore taking into account (13), (14) and (15) we get

$$\max_{M_p < k \leq M_{p+1}, N_q < l \leq N_{q+1}} |S_{k,l}(x, y) - S_{M_p, N_q}(x, y)| = 0$$

almost everywhere on $(0, 1)^2$. Finally taking into account (11) we finished proof of sufficiency.

Necessity. a) Let

$$\sum_{p,q=1}^{\infty} \frac{1}{\omega_1(M_p) \omega_2(N_q)} = \infty.$$

Without loss of generality it can be assumed that

$$\sum_{p=1}^{\infty} \frac{1}{\omega_1(M_p)} = \infty.$$

Then there exist numbers $c_p > 0$ such that

$$\sum_{p,q=1}^{\infty} c_p^2 \omega_1(M_p) < \infty \text{ and } \sum_{p=1}^{\infty} c_p = \infty.$$

Take $a_{M_p, N_1} = c_p$, ($p = 1, 2, \dots$), $a_{mn} = 0$, ($(m, n) \neq (M_p, N_1)$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, $p \in \mathbb{N}$). Let $\varphi_{M_p, N_1}(x, y) = 1$, ($p = 1, 2, \dots$), $(x, y) \in (0, 1)^2$ and choose as other functions an arbitrary ONS orthogonal to 1. The system $\{\varphi_{mn}\}$ is diagonal Δ_k -ONS, for which

$$\sum_{m,n=1}^{\infty} a_{mn} \varphi_{mn}(x, y) = \sum_{p=1}^{\infty} c_p = \infty \quad (x, y) \in (0, 1)^2$$

Though

$$\sum_{m,n=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) = \sum_{p=1}^{\infty} c_p^2 \omega_1(M_p) \omega_2(N_1) < \infty.$$

b) Let condition (4) is not fulfilled. Without loss of generality it can be assumed that the condition $\log^2 m = O(\omega_1(m))$, ($m \rightarrow \infty$) is not fulfilled. Then there exist (see [3, Theorem 1.]), numbers b_m and $(M_p, M_{p+1}]$ -ONS $\{\varphi_m\}$ such that

$$(16) \quad \sum_{m=1}^{\infty} b_m^2 \omega_1(m) < \infty,$$

though

$$(17) \quad \sum_{m=1}^{\infty} b_m \varphi_m(x)$$

diverges a. e. on $(0, 1)$.

Take $a_{m,1} = b_m$, $(m = 1, 2, \dots)$, $a_{mn} = 0$, $(m \in \mathbb{N}, n \geq 2)$. Let $\{\psi_n\}$ be an ONS from $L^2(0, 1)$ such that $\psi_1(y) = 1$, $y \in (0, 1)$. The system $\varphi_{mn}(x, y) = \varphi_m(x)\psi_n(y)$ is a diagonal Δ_k -orthonormal system. Then taking into account (16), (17) we have

$$\sum_{m,n=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) = \omega_2(1) \sum_{m=1}^{\infty} b_m^2 \omega_1(m) < \infty,$$

though the series

$$\sum_{m,n=1}^{\infty} a_{mn} \varphi_{mn}(x, y) = \sum_{m=1}^{\infty} b_m \varphi_m(x) \psi_1(y)$$

diverges a. e. on $(0, 1)^2$. □

Corollary. *If we take $\omega_1(m) = \omega_2(m) = \log^2 m$ then we obtain the following theorem:*

a) *If*

$$(18) \quad \sum_{p,q=1}^{\infty} \frac{1}{\log^2(M_p) \log^2(N_q)} < \infty,$$

then for every diagonal Δ_k -ONS $\{\varphi_{mn}\}$ the condition

$$\sum_{m,n=1}^{\infty} a_{mn}^2 \log^2 m \log^2 n < \infty$$

guarantees the convergence a. e. on $(0, 1)^2$ of the series (1).

b) *If however*

$$\sum_{p,q=1}^{\infty} \frac{1}{\log^2 M_p \log^2 N_q} = \infty,$$

then there exist numbers b_{mn} and diagonal Δ_k -ONS $\{\psi_{mn}\}$ such that the series

$$\sum_{m,n=1}^{\infty} b_{mn} \psi_{mn}(x, y)$$

diverges a. e. on $(0, 1)^2$ though

$$\sum_{m,n=1}^{\infty} b_{mn}^2 \log^2 m \log^2 n < \infty.$$

Remark 1. For example if we take $M_p = [2^{p^\alpha}]$, $N_q = [2^{q^\alpha}]$, $\alpha > \frac{1}{2}$, then the condition (18) is fulfilled. Therefore the two-dimensional analog of Menshov-Rademacher's Theorem (see [4], theorem 1) is fulfilled for any Δ_k -ONS $\{\varphi_{mn}\}$.

If $M_p = [2^{p^\alpha}]$, $N_q = [2^{q^\alpha}]$, $0 < \alpha \leq \frac{1}{2}$, then $\{\log^2 m \log^2 n\}$ will be the Weyl multiplier for the convergence a. e. not for each Δ_k -ONS. From proved

Theorem it follows that in that case $\left\{ \log^{\frac{1}{\alpha}+\varepsilon} m \log^{\frac{1}{\alpha}+\varepsilon} n \right\}$ ($\varepsilon > 0$) is the Weyl multiplier.

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