

APPROXIMATION PROPERTIES OF PARTIAL SUMS OF FOURIER SERIES

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ABSTRACT. In this paper we find class of functions for which the Lebesgue estimate can be improved.

Let $C([0, 2\pi])$ denote the space of continuous function f with period 2π . If $f \in C([0, 2\pi])$ then the function

$$\omega_p(\delta, f) := \sup_x \sup_{|h| \leq \delta} |\Delta_p(x; h, f)|, \omega_1(\delta, f) := \omega(\delta, f)$$

is called the modulus of continuity of the function f , where

$$\Delta_1(x; h, f) := f(x + h) - f(x),$$

$$\Delta_{p+1}(x; h, f) = \Delta_p(x + h; h, f) - \Delta_p(x; h, f).$$

Let $S_n(f, x)$ be the n th partial sums of the trigonometric Fourier series of the function f .

The estimation of Lebesgue (see [3], or [1]) is well-known

$$\|f - S_n(f)\| \leq c\omega\left(\frac{1}{n}, f\right) \log(n+2).$$

Generalization of this estimation were studied by Chanturia [2], Oskolkov [6], Karchava [4, 5]. There arises a question: for what subclasses of $C([0, 2\pi])$ the Lebesgue estimate can be improved?

We prove that the following are true

Theorem 1. *Let $f \in C([0, 2\pi])$. Then*

$$\|f - S_n(f)\|_C \leq c \sum_{k=1}^{[\log n]} \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}.$$

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Corollary 1. *Let $f \in C([0, 2\pi])$. Then*

$$\|f - S_n(f)\|_C \leq c \max_{1 \leq k \leq [\log \log n]} \omega_k\left(\frac{1}{n}, f\right).$$

Corollary 2. *Let $f \in C([0, 2\pi])$ and $\omega_k\left(\frac{1}{n}, f\right) \leq M$. Then*

$$\|f - S_n(f)\|_C \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Corollary 3. *Let $f \in C([0, 2\pi])$ and $\omega_k\left(\frac{1}{n}, f\right) \leq \frac{2^k}{l(k)}$, where*

$$\sum_{k=1}^{\infty} \frac{1}{l(k)} < \infty.$$

Then

$$\|f - S_n(f)\|_C \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Let $T_n(x)$ be Vale-Poisson polynomial which provides best approximation of function f in the space $C([0, 2\pi])$ (see [3]), in particular

$$T_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) V_n(t) dt, \quad \frac{1}{\pi} \int_{-\pi}^{\pi} |V_n(t)| dt \leq 1.$$

Set

$$g(t) := f(x+t) + f(x-t) - 2f(x) - T_n(x+t) - T_n(x-t) + 2T_n(x).$$

It is evident that

$$\omega_p(\delta, T_n) \leq c\omega_p(\delta, f)$$

and

$$(1) \quad \sum_{i=k+1}^n \frac{(-1)^i}{i} = \frac{(-1)^{k+1}}{2k} + P_k + Q_n, \quad P_k \leq \frac{c}{k^2}, \quad Q_n \leq \frac{c}{n}.$$

We write

$$(2) \quad \int_0^{\pi} g(t) D_n(t) dt = \int_0^{\pi} g(t) \left(D_n(t) - \frac{\sin nt}{t} \right) dt + \int_0^{\pi} g(t) \frac{\sin nt}{t} dt.$$

Since

$$D_n(t) - \frac{\sin nt}{t}$$

is bounded function and

$$|g(t)| \leq 4E_n(f).$$

Hence

$$(3) \quad \left| \int_0^{\pi} g(t) \left(D_n(t) - \frac{\sin nt}{t} \right) dt \right| \leq cE_n(f),$$

where $E_n(f)$ is best approximation of the function f .

On the other hand,

$$(4) \quad \left| \int_0^{\pi/n} g(t) \frac{\sin nt}{t} dt \right| \leq \|g\|_c \int_0^{\pi/n} \frac{\sin nt}{t} dt \leq cE_n(f).$$

Combining (2)-(4) we have

$$(5) \quad \int_0^{\pi} g(t) D_n(t) dt = \int_0^{\pi} g(t) \frac{\sin nt}{t} dt + \gamma,$$

where

$$|\gamma| \leq cE_n(f).$$

It is evident that

$$(6) \quad \begin{aligned} \left| \int_{\pi/n}^{\pi} g(t) \frac{\sin nt}{t} dt \right| &= \left| \sum_{k=1}^{n-1} \int_{\pi k/n}^{\pi(k+1)/n} g(t) \frac{\sin nt}{t} dt \right| \\ &= \left| \sum_{k=1}^{n-1} \int_0^{\pi/n} g\left(t + \frac{\pi k}{n}\right) \frac{(-1)^k \sin nt}{t + \pi k/n} dt \right| \\ &= \left| \sum_{k=1}^{n-1} \int_0^{\pi} g\left(\frac{u + \pi k}{n}\right) \frac{(-1)^k \sin u}{u + \pi k} du \right| \\ &\leq \pi \max_{0 \leq u \leq \pi} \left| \sum_{k=1}^{n-1} g\left(\frac{u + \pi k}{n}\right) \frac{(-1)^k \sin u}{u + \pi k} \right| \\ &\leq \pi \left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \frac{(-1)^k}{u_0 + \pi k} \right| \\ &\leq \pi \left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \left(\frac{(-1)^k}{u_0 + \pi k} - \frac{(-1)^k}{\pi k} \right) \right| \\ &\quad + \pi \left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \frac{(-1)^k}{\pi k} \right| \\ &= I + II. \end{aligned}$$

where

$$\left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \frac{(-1)^k}{u_0 + \pi k} \right| = \sup_u \left| \sum_{k=1}^{n-1} g\left(\frac{u + \pi k}{n}\right) \frac{(-1)^k}{u + \pi k} \right|.$$

It is clear that

$$(7) \quad I \leq \|g\|_c \sum_{k=1}^{n-1} \frac{1}{k^2} = O(E_n(f)).$$

Set

$$u_k := \frac{u_0 + \pi k}{n}.$$

Using Abel transformation and (1), for II we obtain

$$\begin{aligned} II &= \left| \sum_{k=1}^{n-1} g(u_k) \frac{(-1)^k}{\pi k} \right| \\ &\leq \left| \sum_{k=1}^{n-2} (g(u_{k+1}) - g(u_k)) \sum_{i=k+1}^{n-1} \frac{(-1)^i}{i} \right| + \left| g(u_1) \sum_{i=1}^{n-1} \frac{(-1)^i}{i} \right| \\ &\leq \left| \sum_{k=1}^{n-2} (g(u_{k+1}) - g(u_k)) \left(\frac{(-1)^{k+1}}{2k} + P_k + Q_n \right) \right| + \left| g(u_1) \sum_{i=1}^{n-1} \frac{(-1)^i}{i} \right| \\ &\leq \left| \sum_{k=1}^{n-2} (g(u_{k+1}) - g(u_k)) \frac{(-1)^{k+1}}{2k} \right| + O(E_n(f)) \\ &= \left| \sum_{k=1}^{n-2} \Delta_1 \left(u_k; \frac{\pi}{n}, g \right) \frac{(-1)^{k+1}}{2k} \right| + O(E_n(f)) \\ &\leq \frac{1}{2} \left| \sum_{k=1}^{n-2} \Delta_1 \left(u_k; \frac{\pi}{n}, g \right) \frac{(-1)^{k+1}}{k} \right| + \gamma_1 \\ &\leq \frac{1}{2^2} \left| \sum_{k=1}^{n-3} \Delta_2 \left(u_k; \frac{\pi}{n}, g \right) \frac{(-1)^{k+1}}{k} \right| + \gamma_1 + \gamma_2 \\ &\leq \cdots \leq \frac{1}{2^p} \left| \sum_{k=1}^{n-3} \Delta_p \left(u_k; \frac{\pi}{n}, g \right) \frac{(-1)^{k+1}}{2k} \right| + \sum_{k=1}^p \gamma_k \end{aligned}$$

where

$$\gamma_k < \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}.$$

Consequently,

$$(8) \quad II \leq \frac{\omega_p\left(\frac{1}{n}, f\right)}{2^p} \log n + \sum_{k=1}^p \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}.$$

Since

$$\frac{\omega_i\left(\frac{1}{n}, f\right)}{2^i} \leq \frac{\omega_j\left(\frac{1}{n}, f\right)}{2^j} \quad (i \geq j),$$

we can write

$$(9) \quad \frac{\omega_p\left(\frac{1}{n}, f\right) p}{2^p} \leq \sum_{k=1}^p \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}.$$

Set $p = [\log n]$. Then from (8) and (9) we obtain

$$(10) \quad II \leq \sum_{k=1}^{[\log n]} \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}.$$

Combining (6), (7) and (10) complete the proof of Theorem 1. $\square$

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