

RICCI SOLITONS IN LORENTZIAN α -SASAKIAN MANIFOLDS

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ABSTRACT. We study Ricci solitons in Lorentzian α -Sasakian manifolds. It is shown that a symmetric parallel second order covariant tensor in a Lorentzian α -Sasakian manifold is a constant multiple of the metric tensor. Using this it is shown that if $\mathcal{L}_V g + 2S$ is parallel, V is a given vector field then (g, V) is Ricci soliton. Further, by virtue of this result Ricci solitons for $(2n + 1)$ -dimensional Lorentzian α -Sasakian manifolds are obtained. Next, Ricci solitons for 3-dimensional Lorentzian α -Sasakian manifold whose scalar curvature is constant are obtained.

1. INTRODUCTION

Ricci flow is an excellent tool for simplifying the structure of a manifold and smooth out the topology of that manifold to make it look more symmetric. It is defined for Riemannian manifolds of any dimension. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the regularity in the metric. It is given by

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric} g.$$

For example, if $ds^2 = e^{2p(x,y)}(dx^2 + dy^2)$, then to compute the Ricci tensor and Laplace-Beltrami operator for two dimensional Riemannian manifold we use the differential forms method of Elie Cartan. We obtain an expression for the Ricci flow:

$$\frac{\partial p}{\partial t} = \Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}.$$

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This is manifestly analogous to the best known of all diffusion equations, the heat equation that is,

$$\frac{\partial T}{\partial t} = \Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},$$

where now $\Delta = D_x^2 + D_y^2$ is the usual Laplacian on the Euclidean plane.

Let $X(t)$ be a time dependent family of smooth vector fields on M generated by a family of diffeomorphisms $\{\phi_t : t \in R\}$ that is one parameter group of transformations, then the relation between $f : M \rightarrow R$ and $\{\phi_t : t \in R\}$ is

$$X(\phi_t(p))f = \frac{df \circ \phi_t}{dt}(p).$$

Let $\sigma(t)$ be a smooth function of time. Since $\phi_t : M \rightarrow M$ is a diffeomorphism and $g(t)$ is a Riemannian metric on M (codomain) then by definition of pull back $\phi_t^*g(t)$ is a metric on M (domain).

Set $\tilde{g}(t) = \sigma(t)\phi_t^*(g(t))$ then we have [21]

$$(1.1) \quad \frac{\partial \tilde{g}}{\partial t} = \sigma'(t)\phi_t^*(g(t)) + \sigma(t)\phi_t^* \frac{\partial g}{\partial t} + \sigma(t)\phi_t^*(L_X g).$$

Suppose we have a metric g_0 , a vector field Y and $\lambda \in R$ (all independent of time) such that

$$(1.2) \quad \mathcal{L}_Y g_0 + 2 \operatorname{Ric} g_0 + 2\lambda g_0 = 0.$$

If we choose $g(t) = g_0$, $\sigma(t) = 1 - 2\lambda t$ and $X(t) = \frac{1}{\sigma(t)}Y$ which gives a family of diffeomorphisms ϕ_t with ϕ_0 identity then using (1.2) in (1.1) $\tilde{g}$ defined above is a Ricci flow with $g(0) = g_0$ that is

$$(1.3) \quad \frac{\partial \tilde{g}}{\partial t} = -2 \operatorname{Ric} \tilde{g}.$$

Hence $\mathcal{L}_X g_0 + 2 \operatorname{Ric} g_0 + 2\lambda g_0 = 0$ is a solution of the Ricci flow and is known as Ricci soliton.

Hereafter, we use the notation S instead of Ric for Ricci tensor.

Thus a Ricci soliton on a Riemannian manifold is defined by

$$(1.4) \quad \mathcal{L}_X g + 2S + 2\lambda g = 0.$$

It is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

An η -Ricci soliton introduced in the paper [3] as a data (g, V, λ, μ) :

$$(1.5) \quad \mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0.$$

1.1. Example (Hamilton Cigar Soliton). Let $M = R^2$ and $\phi_t : R^2 \rightarrow R^2$ defined by $\phi_t(x, y) = (e^{-2t}x, e^{-2t}y)$ forms a family of one parameter group of diffeomorphisms. The vector field X generated by $\{\phi_t\}$ is $X = -2 \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)$. The metric g_0 is obtained as $g_0 = \frac{dx^2 + dy^2}{1+x^2+y^2}$, $\tilde{g}(t) = \phi_t^*(g_0) = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}$, $\operatorname{Ric} g_0 = \frac{2}{1+x^2+y^2}g_0$, $\mathcal{L}_X g_0 = \frac{4}{1+x^2+y^2}g_0$. Using (1.4) we have $\lambda = 0$. Hence this Ricci

soliton is steady and is called cigar soliton because it is asymptotic to a flat cylinder at infinity.

In 1923, Eisenhart [7] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1925, Levy [12] has obtained the necessary and sufficient conditions for the existence of such tensors. Recently Sharma [9] and [19] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non singular) tensor on an n -dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [16] that on a Sasakian manifold there is no nonzero parallel 2-form. In 1964, Y. Wong [23] proved that the existence of linear connections w.r.t which given tensor fields are parallel or recurrent. Also the parallelism of h is involved and appears in his paper as the theory of totally geodesic maps, and $\nabla h = 0$ is equivalent with the fact that $I: (M, g) \rightarrow (M, h)$ is a totally geodesic map. In 2007, Lovejoy Das [5] in his paper proved that a second order symmetric parallel tensor on an α -K-contact ($\alpha \in R_0$) manifold is a constant multiple of the associated metric tensor and he also proved that there is no nonzero skew symmetric second order parallel tensor on an α -Sasakian manifold.

Constantin Calin and Mircea Crasmareanu [2] have extended the Eisenhart problem to Ricci solitons in f -Kenmotsu manifolds. They have studied the case of f -Kenmotsu manifolds satisfying a special condition called regular and show that a symmetric parallel tensor field of second order is a constant multiple of the Riemannian metric. Using this result they have obtained results on Ricci solitons concerned to f -Kenmotsu manifolds and 3-dimensional β -Kenmotsu manifolds.

2. BASIC CONCEPTS OF LORENTZIAN α -SASAKIAN MANIFOLDS

A differentiable manifold of dimension $(2n + 1)$ is called Lorentzian α -Sasakian manifold if it admits a $(1, 1)$ tensor field ϕ , a vector field ξ and 1-form η and Lorentzian metric g which satisfy on M respectively such that,

$$(2.1) \quad \phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0,$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

$$(2.3) \quad \nabla_X \xi = \alpha \phi X, \quad (\nabla_X \eta)Y = \alpha g(\phi X, Y),$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g on M .

Further, on an Lorentzian α -Sasakian manifold M the following relations hold:

$$(2.4) \quad R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y],$$

$$(2.5) \quad R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X],$$

$$(2.6) \quad S(X, \xi) = 2n\alpha^2\eta(X),$$

$$(2.7) \quad Q\xi = 2n\alpha^2\xi,$$

$$(2.8) \quad S(\xi, \xi) = -2n\alpha^2,$$

where α is some constant, R is the Riemannian curvature, S is the Ricci curvature and Q is the Ricci operator given by $S(X, Y) = g(QX, Y)$.

2.1. Example. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard co-ordinates in R^3 . Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$(2.9) \quad E_1 = e^z \frac{\partial}{\partial y}, \quad E_2 = e^z \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad E_3 = k \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_2) &= g(E_2, E_3) = g(E_1, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1, \end{aligned}$$

where g is given by

$$g = \frac{1}{e^{2z}} [dx \otimes dx + dy \otimes dy] - \frac{1}{k^2} dz \otimes dz.$$

The (ϕ, ξ, η) is given by

$$\eta = \frac{1}{k} dz, \quad \xi = E_3 = k \frac{\partial}{\partial z}, \quad \phi E_1 = -E_1, \quad \phi E_2 = -E_2, \quad \phi E_3 = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = -1, \quad \phi^2 U = U + \eta(U)E_3, \quad g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W),$$

for any vector fields U, W on M . By definition of Lie bracket, we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = -kE_1, \quad [E_2, E_3] = -kE_2.$$

Let ∇ be Levi-Civita connection with respect to the above metric g given by Koszul formula

$$(2.10) \quad \begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then

$$(2.11) \quad \begin{aligned} \nabla_{E_1} E_1 &= -kE_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -kE_1, \\ \nabla_{E_2} E_1 &= 0, \quad \nabla_{E_2} E_2 = -kE_3, \quad \nabla_{E_2} E_3 = -kE_2, \\ \nabla_{E_3} E_1 &= 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0. \end{aligned}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3 , that is $X = \sum_{i=1}^3 a_i E_i$ and $Y = \sum_{i=1}^3 b_i E_i$ where $a_i, b_i (i = 1, 2, 3)$ are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy equations (2.1), (2.2) and (2.3) with $\alpha = k$. Thus M is a Lorentzian α -Sasakian manifold.

Definition 1. Let M be a Riemannian manifold with metric g , ξ an unitary vector field, η the 1-form dual to ξ . Further, let h a symmetric tensor field of $(0, 2)$ -type on M which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [16]

$$(2.12) \quad \nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0,$$

we obtain the relation [16]:

$$(2.13) \quad h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0,$$

then by taking $Z = W = \xi$ in (2.13) it reduces to

$$(2.14) \quad A[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] = 0,$$

where $A \neq 0$ is some scalar function then M is called regular (that is $M_A^{(2n+1)}(\xi)$ is called regular if $A \neq 0$).

3. PARALLEL SYMMETRIC SECOND ORDER TENSORS AND RICCI SOLITONS IN LORENTZIAN α -SASAKIAN MANIFOLDS

Fix h a symmetric tensor field of $(0, 2)$ -type which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [16] in (2.12) we obtain (2.13). Replacing $Z = W = \xi$ in (2.13) and using (2.4) and by the symmetry of h , we have

$$(3.1) \quad 2\alpha^2[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] = 0.$$

Put $X = \xi$ in (3.1), we have

$$(3.2) \quad 2\alpha^2[\eta(Y)h(\xi, \xi) + h(Y, \xi)] = 0.$$

Since $2\alpha^2 \neq 0$, by definition (1) Lorentzian α -Sasakian manifold is regular.

By (3.2), we have

$$(3.3) \quad h(Y, \xi) = -\eta(Y)h(\xi, \xi).$$

Differentiating (3.3) covariantly with respect to X , we have

$$(3.4) \quad \begin{aligned} (\nabla_X h)(Y, \xi) + h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) = \\ - [(\nabla_X \eta)(Y) + \eta(\nabla_X Y)]h(\xi, \xi) \\ - \eta(Y)[(\nabla_X h)(\xi, \xi) + 2h(\nabla_X \xi, \xi)]. \end{aligned}$$

By using (2.2), (2.3) and (3.3), we have

$$(3.5) \quad -h(Y, \phi X) = g(Y, \phi X)h(\xi, \xi),$$

we deduce the above equation then we have

$$(3.6) \quad h(X, Y) = -g(X, Y)h(\xi, \xi),$$

which together with the standard fact that the parallelism of h implies the $h(\xi, \xi)$ is a constant and via (3.3) yields the following:

Theorem 3.1. *A symmetric parallel second order covariant tensor in a regular Lorentzian α -Sasakian manifolds is a constant multiple of the metric tensor.*

Corollary 1. *A locally Ricci symmetric ($\nabla S = 0$) regular Lorentzian α -Sasakian manifolds is an Einstein manifold.*

Remark: The following statements for Lorentzian α -Sasakian manifolds are equivalent. The manifold is

- (i) Einstein
- (ii) locally Ricci symmetric
- (iii) Ricci semi-symmetric that is $R \cdot S = 0$.

The implication (i) $\implies$ (ii) $\implies$ (iii) is trivial. Now we prove the implication (iii) $\implies$ (i) and $R \cdot S = 0$ means exactly (2.13) with replaced h by S that is

$$(3.7) \quad (R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V).$$

Considering $R \cdot S = 0$ and putting $X = \xi$ in equation (3.7), we have

$$(3.8) \quad S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0.$$

By using (2.5) and (2.6), we obtain

$$(3.9) \quad 2n\alpha^4 g(Y, U)\eta(V) - \alpha^2 \eta(U)S(Y, V) + 2n\alpha^4 g(Y, V)\eta(U) - \alpha^2 \eta(V)S(U, Y) = 0.$$

Again by putting $U = \xi$ in the above equation and by using (2.1), (2.2) and (2.6), we obtain

$$(3.10) \quad S(Y, V) = 2n\alpha^2 g(Y, V).$$

In conclusion:

Proposition 1. *A Ricci semi-symmetric regular Lorentzian α -Sasakian manifolds is Einstein.*

We close this section with applications of our Theorem to Ricci solitons:

Corollary 2. *Suppose that on a regular Lorentzian α -Sasakian manifolds the $(0, 2)$ -type field $\mathcal{L}_V g + 2S$ is parallel where V is a given vector field. Then (g, V) yield a Ricci soliton. In particular, if the given regular Lorentzian α -Sasakian manifold is Ricci-semi symmetric with $\mathcal{L}_V g$ parallel, we have the same conclusion.*

Proof. Follows from Theorem 3.1 and Corollary 1. □

Naturally, two situations appear regarding the vector field V : $V \in \text{Span } \xi$ and $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$.

We are interested in expressions for $\mathcal{L}_\xi g + 2S$. A straightforward computation gives

$$(3.11) \quad (\mathcal{L}_\xi g)(X, Y) = 2\alpha g(\phi X, Y).$$

The metric g is called η -Einstein if there exists two real functions a and b such that the Ricci tensor of g is

$$(3.12) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y).$$

Let $e_i = 1, 2, \dots, (2n+1)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (3.12) and taking summation over i then we get

$$(3.13) \quad r = (2n+1)a - b.$$

Again putting $X = Y = \xi$ in (3.12) then by using (2.1), (2.2) and (2.8), we have

$$(3.14) \quad -a + b = -2n\alpha^2,$$

from (3.13) and (3.14), we obtain the values of a and b

$$a = \frac{r}{2n} - \alpha^2, \quad b = \frac{r}{2n} - (2n+1)\alpha^2.$$

Substituting the values of a and b in (3.12), we have

$$(3.15) \quad S(X, Y) = \left[\frac{r}{2n} - \alpha^2 \right] g(X, Y) + \left[\frac{r}{2n} - (2n+1)\alpha^2 \right] \eta(X)\eta(Y).$$

The above equation shows that Lorentzian α -Sasakian manifold is η -Einstein.

For $(2n+1)$ -dimensional Lorentzian α -Sasakian manifolds, we have

$$(3.16) \quad h(X, Y) = (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y).$$

Then in (3.16) substituting the values of (3.11) and (3.15), we have

$$(3.17) \quad h(X, Y) = 2\alpha g(\phi X, Y) + \left[\frac{r}{n} - 2\alpha^2 \right] g(X, Y) + \left[\frac{r}{n} - 2(2n+1)\alpha^2 \right] \eta(X)\eta(Y).$$

Differentiating the above equation (3.17) with respect to Z then we have

$$(3.18) \quad (\nabla_Z h)(X, Y) = 2(Z\alpha)g(\phi X, Y) + \left[\frac{\nabla_Z r}{n} - 4\alpha(Z\alpha) \right] g(X, Y) + \left[\frac{\nabla_Z r}{n} - 4(2n+1)\alpha(Z\alpha) \right] \eta(X)\eta(Y) + 2\alpha g((\nabla_Z \phi)X, Y) + \left[\frac{r}{n} - 2(2n+1)\alpha^2 \right] \{ \alpha g(X, \phi Z)\eta(Y) + \alpha g(Y, \phi Z)\eta(X) \},$$

by substituting $Z = \xi$ and $X = Y \in (Span \xi)^\perp$ in the above equation, we have

$$(3.19) \quad \nabla_\xi r = 0,$$

provided h is parallel. Thus r is constant scalar, then we state that:

Proposition 2. *An η -Einstein Lorentzian α -Sasakian Ricci soliton (g, ξ, λ) with constant scalar curvature r is shrinking.*

Proof. From equation (1.4) and (3.16), we have

$$h(X, Y) = -2\lambda g(X, Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$(3.20) \quad h(\xi, \xi) = 2\lambda.$$

Now considering (3.17), that is

$$h(X, Y) = 2\alpha g(\phi X, Y) + \left[\frac{r}{n} - 2\alpha^2\right] g(X, Y) + \left[\frac{r}{n} - 2(2n+1)\alpha^2\right] \eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$(3.21) \quad h(\xi, \xi) = -4n\alpha^2.$$

By equating (3.20) and (3.21), we have

$$(3.22) \quad \lambda = -2n\alpha^2.$$

This shows that $\lambda < 0$ that is the Ricci soliton in $(2n+1)$ -dimensional Lorentzian α -Sasakian is shrinking. $\square$

We compute an expression for Ricci tensor for 3-dimensional Lorentzian α -Sasakian manifold as follows: The curvature tensor for 3-dimensional Riemannian manifold is given by

$$(3.23) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \\ - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y],$$

put $Z = \xi$ in the above equation that is in (3.23) and by using (2.2), (2.4) and (2.6), we obtain

$$(3.24) \quad \left[\frac{r}{2} - \alpha^2\right] [\eta(Y)X - \eta(X)Y] = \eta(Y)QX - \eta(X)QY.$$

Again put $Y = \xi$ in the equation (3.24) and by using (2.1) and (2.7), we have

$$(3.25) \quad QX = \left[\frac{r}{2} - \alpha^2\right] X + \left[\frac{r}{2} - 3\alpha^2\right] \eta(X)\xi$$

and

$$(3.26) \quad S(X, Y) = \left[\frac{r}{2} - \alpha^2\right] g(X, Y) + \left[\frac{r}{2} - 3\alpha^2\right] \eta(X)\eta(Y),$$

where r is the scalar curvature and α is a constant.

For a 3-dimensional Lorentzian α -Sasakian manifolds, we obtain

$$(3.27) \quad h(X, Y) = (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y).$$

By using (3.11) and (3.26) in (3.27), we have

$$(3.28) \quad h(X, Y) = 2\alpha g(\phi X, Y) + [r - 2\alpha^2]g(X, Y) + [r - 6\alpha^2]\eta(X)\eta(Y).$$

Differentiating the above equation with respect to Z then we have

$$(3.29) \quad (\nabla_Z h)(X, Y) = \\ 2(Z\alpha)g(\phi X, Y) + 2\alpha g((\nabla_Z \phi)X, Y) + [\nabla_Z r - 4\alpha(Z\alpha)]g(X, Y)$$

$$+ [\nabla_Z r - 6(2\alpha(Z\alpha))] \eta(X) \eta(Y) + (r - 6\alpha^2) [\alpha g(X, \phi Z) \eta(Y) + \alpha g(Y, \phi Z) \eta(X)].$$

Substituting $Z = \xi$, $X = Y \in (\text{Span } \xi)^\perp$ in (3.29) then we have

$$(3.30) \quad \nabla_\xi r = 0,$$

provided h is parallel. Thus r is constant scalar, then we state that:

Proposition 3. *A Ricci soliton (g, ξ, λ) in 3-dimensional Lorentzian α -Sasakian manifold with constant scalar curvature r is shrinking.*

Proof. From equation (1.4) and (3.27), we have

$$h(X, Y) = -2\lambda g(X, Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$(3.31) \quad h(\xi, \xi) = 2\lambda.$$

Now considering (3.28), that is

$$h(X, Y) = 2\alpha g(\phi X, Y) + [r - 2\alpha^2]g(X, Y) + [r - 6\alpha^2]\eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$(3.32) \quad h(\xi, \xi) = -4\alpha^2.$$

By equating (3.31) and (3.32), we have

$$(3.33) \quad \lambda = -2\alpha^2.$$

This shows that $\lambda < 0$ that is the Ricci soliton in 3-dimensional Lorentzian α -Sasakian is shrinking. $\square$

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