

RICCI CURVATURE OF QUATERNION SLANT SUBMANIFOLDS IN QUATERNION SPACE FORMS

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ABSTRACT. In this article, we obtain sharp estimate of the Ricci curvature of quaternion slant, bi-slant and semi-slant submanifolds in a quaternion space form, in terms of the squared mean curvature.

1. INTRODUCTION

In [15], S. Ishihara defined a quaternion manifold (or quaternion Kaehlerian manifold) as a Riemannian manifold whose holonomy group is a subgroup of $\mathbf{Sp}(1)$. It is well known that on a quaternion manifold $\tilde{M}$, there exists a 3-dimensional vector bundle E of tensors of type $(1, 1)$ with local cross-section of almost Hermitian structures satisfying certain conditions [4]. A submanifold M in a quaternion manifold $\tilde{M}$ is called a quaternion submanifold if each tangent space of M is carried into itself by each section of E . In [3] authors studied quaternion CR-submanifolds of quaternion manifolds. A quaternion manifold is a quaternion space form if its quaternion sectional curvatures are constant. In [17] authors established a sharp relationship between the Ricci curvature and squared mean curvature of a quaternion CR-submanifold in a quaternion space form. Slant submanifolds of Kaehler manifolds were defined by B. Y. Chen [10] and studied by several geometers [20, 23].

On the other hand, N. Papaghiuc [18] introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifolds which include proper CR-submanifolds and proper slant submanifolds as particular cases. The purpose of present paper is to study quaternion slant, bi-slant and semi-slant submanifolds in a quaternion space form.

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2. PRELIMINARIES

Let $\tilde{M}$ be a $4m$ -dimensional Riemannian manifold with metric tensor g . Then $\tilde{M}$ is said to be a quaternion Kaehlerian manifold, if there exists a 3-dimensional vector bundle E consisting of tensors of type $(1, 1)$ with local basis of almost Hermitian structures J_1, J_2 and J_3 such that

(a)

$$\begin{aligned} J_1^2 &= -I, \quad J_2^2 = -I, \quad J_3^2 = -I, \\ J_1 J_2 &= -J_2 J_1 = J_3, \quad J_2 J_3 = -J_3 J_2 = J_1, \quad J_3 J_1 = -J_1 J_3 = J_2, \end{aligned}$$

where I denotes the identity tensor field of type $(1, 1)$ on $\tilde{M}$.

(b) for any local cross-section J of E and any vector X tangent to $\tilde{M}$, $\tilde{\nabla}_X J$ is also a local cross-section of E , where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$.

The condition (b) is equivalent to the following condition:

(c) there exist local 1-forms p, q and r such that

$$\begin{aligned} \tilde{\nabla}_X J_1 &= r(X)J_2 - q(X)J_3, \\ \tilde{\nabla}_X J_2 &= -r(X)J_1 + p(X)J_3, \\ \tilde{\nabla}_X J_3 &= q(X)J_1 - p(X)J_2. \end{aligned}$$

Now, let X be an unit vector tangent to the quaternion manifold $\tilde{M}$, then $X, J_1 X, J_2 X$ and $J_3 X$ form an orthonormal frame. We denote by $Q(X)$ the 4-plane spanned by them and call $Q(X)$ the quaternion section determined by X . For any orthonormal vectors X, Y tangent to $\tilde{M}$, the plane $X \wedge Y$ spanned by X, Y is said to be totally real if $Q(X)$ and $Q(Y)$ are orthogonal. Any plane in a quaternion section is called a quaternion plane. The sectional curvature of a quaternion plane is called a quaternion sectional curvature. A quaternion manifold is called a quaternion space form if its quaternion sectional curvatures are equal to a constant.

Let $\tilde{M}(c)$ be a $4m$ -dimensional quaternion space form of constant quaternion sectional curvature c . The curvature tensor of $\tilde{M}(c)$ has the following expression ([15]):

$$\begin{aligned} (2.1) \quad \tilde{R}(X, Y)Z &= \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y \\ &\quad + g(J_1 Y, Z)J_1 X - g(J_1 X, Z)J_1 Y + 2g(X, J_1 Y)J_1 Z \\ &\quad + g(J_2 Y, Z)J_2 X - g(J_2 X, Z)J_2 Y + 2g(X, J_2 Y)J_2 Z \\ &\quad + g(J_3 Y, Z)J_3 X - g(J_3 X, Z)J_3 Y + 2g(X, J_3 Y)J_3 Z \}, \end{aligned}$$

for any vector fields X, Y, Z tangent to $\tilde{M}$. The equation (2.1) can be written as:

$$(2.2) \quad \tilde{R}(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ + \sum_{i=1}^3 [g(J_i Y, Z)J_i X - g(J_i X, Z)J_i Y + 2g(X, J_i Y)J_i Z]\}$$

for any vector fields X, Y, Z tangent to $\tilde{M}$.

Now, we recall

Definition 2.1 ([3]). Let M be a Riemannian manifold isometrically immersed in a quaternion manifold $\tilde{M}$. A distribution $D : p \rightarrow D_p \subseteq T_p M$ is called a *quaternion distribution* if we have $J_i(D) \subseteq D$, $i = 1, 2, 3$. In other words, D is a quaternion distribution if D is carried into itself by its quaternion structure.

Definition 2.2 ([3]). A submanifold M in a quaternion manifold $\tilde{M}$ is called a *quaternion CR-submanifold* if it admits a differentiable quaternion distribution D such that its orthogonal complementary distribution $D^\perp$ is totally real, i.e., $J_i(D_p^\perp) \subseteq T_p^\perp M$ and D is invariant under quaternion structure, that is, $J_i(D_p) \subseteq D_p$, $i = 1, 2, 3$, for any $p \in M$, where $T_p^\perp M$ denotes the normal space of M in $\tilde{M}$ at p .

A submanifold M of a quaternion manifold $\tilde{M}$ is called a *quaternion submanifold* if $\dim D_p^\perp = 0$ and a *totally real submanifold* if $\dim D_p = 0$. A quaternion CR-submanifold is said to be *proper* if it is neither totally real nor quaternionic.

Definition 2.3 ([10]). A submanifold M of a quaternion space form $\tilde{M}(c)$ is said to be *quaternion slant submanifold* if for any $p \in M$ and any $X \in T_p M$, the angle between $J_i(X)$, $i = 1, 2, 3$ and $T_p M$ is a constant $\theta \in [0, \frac{\pi}{2}]$, called the slant angle of quaternion submanifold M in $\tilde{M}(c)$.

In particular, quaternion submanifolds and *totally real* submanifolds of $\tilde{M}(c)$ are quaternion slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

Definition 2.4 ([18]). A submanifold M of a quaternion space form $\tilde{M}(c)$ is called a *quaternion bi-slant submanifold* if there exist two orthogonal distributions D_1 and D_2 on M such that

- (i) TM admits orthogonal direct decomposition, i.e., $TM = D_1 \oplus D_2$.
- (ii) For any $i = 1, 2$, the distribution D_i is slant distribution with slant angle θ_i .

Let $4d_1 = \dim D_1$ and $4d_2 = \dim D_2$. If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus slant submanifolds are particular cases of bi-slant submanifolds.

Definition 2.5 ([18]). Let M be a submanifold of a quaternion space form $\tilde{M}(c)$, then we say that M is a *semi-slant submanifold* if there exist two orthogonal distributions D_1 and D_2 on M such that

- (i) TM admits orthogonal direct decomposition, i.e., $TM = D_1 \oplus D_2$.
- (ii) The distribution D_1 is invariant by J_i , $i = 1, 2, 3$, i.e., $J_i(D_1) = D_1$.
- (iii) The distribution D_2 is slant with respect to J_1, J_2, J_3 with slant angle $\theta \neq 0$, i.e. for any non-zero vector $X \in D_2(p)$, $p \in M$, the angle between $J_i X$, $i = 1, 2, 3$ and tangent subspace $D_2(p)$ is constant, that is, it is independent of the choice of $p \in M$ and $X \in D_2(p)$.

Now, we also recall the following Lemma of Chen [11].

Lemma 2.1 ([11]). Let $a_1, \dots, a_n, b$ be $(n+1), n \geq 2$ real numbers such that

$$\left(\sum_{i=1}^n a_i \right)^2 = (n-1) \left(\sum_{i=1}^n a_i^2 + b \right).$$

Then $2a_1a_2 \geq b$ with equality holding if and only if

$$a_1 + a_2 = a_3 = \dots = a_n.$$

Let M be a submanifold of a quaternion space form $\tilde{M}(c)$. We denote by g the metric tensor of $\tilde{M}(c)$ as well as that induced on M . Let ∇ be the induced connection on M . The Gauss and Weingarten formulae for M are given respectively by

$$(2.3) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$(2.4) \quad \tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V$$

for any vector fields X, Y tangent to M and any vector field V normal to M , where h , A_V and $\nabla^\perp$ are the second fundamental form, the shape operator in the direction of V and the normal connection induced by ∇ on the normal bundle $T^\perp M$ respectively. The second fundamental form and the shape operator are related by

$$(2.5) \quad g(h(X, Y), V) = g(A_V X, Y).$$

For the second fundamental form h , we define the covariant differentiation $\tilde{\nabla}$ with respect to the connection in $TM \oplus T^\perp M$ by

$$(2.6) \quad (\tilde{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for any vector fields X, Y, Z tangent to M .

The Gauss, Codazzi and Ricci equations for M are given by

$$(2.7) \quad R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

$$(2.8) \quad (R(X, Y), Z)^\perp = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z),$$

$$(2.9) \quad \tilde{R}(X, Y, V, \eta) = R^\perp(X, Y, V, \eta) - g([A_V, A_\eta]X, Y),$$

for any vector fields X, Y, Z, W tangent to M and V, η normal to M , where R and $R^\perp$ are the curvature tensors with respect to ∇ and $\nabla^\perp$ respectively.

The mean curvature vector $H(p)$ at $p \in M$ is defined by

$$(2.10) \quad H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i),$$

where n denotes the dimension of M . If, we have

$$(2.11) \quad h(X, Y) = \lambda g(X, Y)H,$$

for any vector fields X, Y tangent to M , then M is called totally umbilical submanifold. In particular, if $h = 0$ identically, M is called a totally geodesic submanifold.

We set

$$(2.12) \quad h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, 4m\}$$

and

$$(2.13) \quad \|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).$$

For any $p \in M$ and X tangent to M , we put

$$(2.14) \quad J_i X = P_i X + T_i X, \quad i = 1, 2, 3$$

where $P_i X$ and $T_i X$ are the tangential and normal components of $J_i X$, respectively.

We recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point $p \in M$ is defined by

$$N_p = \{X \in T_p M \mid h(X, Y) = 0 \text{ for all } Y \in T_p M\}.$$

3. QUATERNION SLANT SUBMANIFOLDS

In this section, we estimate the Ricci curvature of quaternion slant, bi-slant and semi-slant submanifolds of a quaternion space form.

Theorem 3.1. *Let M be an n -dimensional quaternion slant submanifold of a $4m$ -dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c . Then*

(I) *For each unit vector $X \in T_p M$, we have*

$$(3.1) \quad \text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (n-1)c + 6c \cos^2 \theta\}.$$

(II) *If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.1) if and only if X belongs to the relative null space N_p .*

Proof. Let $p \in M$, we choose an orthonormal basis $\{e_1, \dots, e_n\}$ for $T_p M$ and $\{e_{n+1}, \dots, e_{4m}\}$ for the normal space $T_p^\perp M$ at p such that $e_n = X$ and e_{n+1} is parallel to the mean curvature vector $\tilde{H}(p)$.

Let M be a quaternion slant submanifold of a $4m$ -dimensional quaternion space form $\tilde{M}(c)$. Then using (2.2) and (2.14) in the equation of Gauss, we have

$$(3.2) \quad R(X, Y, Z, W) = \frac{c}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ + \sum_{i=1}^3 [g(P_i Y, Z)g(P_i X, W) - g(P_i X, Z)g(P_i Y, W) \\ + 2g(X, P_i Y)g(P_i Z, W)]\} \\ + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W))$$

for any vector fields X, Y, Z, W tangent to M .

Let $p \in M$ and an orthonormal basis $\{e_1, \dots, e_n = X\}$ in $T_p M$. The Ricci tensor $S(X, Y)$ is given by

$$(3.3) \quad S(X, Y) = \sum_{j=1}^n R(e_j, X, Y, e_j) \\ = \frac{c}{4} \{g(X, Y)g(e_j, e_j) - g(e_j, Y)g(X, e_j) \\ + \sum_{i=1}^3 [g(P_i X, Y)g(P_i e_j, e_j) - g(P_i e_j, Y)g(P_i X, e_j) \\ + 2g(e_j, P_i X)g(P_i Y, e_j)]\} \\ + g(h(e_j, e_j), h(X, Y)) - g(h(e_j, Y), h(X, e_j)) \\ = \frac{c}{4} \{(n-1)g(X, Y) + 3 \sum_{i=1}^3 g(P_i X, P_i Y)\} \\ + \sum_{j=1}^n \{g(h(e_j, e_j), h(X, Y)) - g(h(e_j, Y), h(X, e_j))\}.$$

The scalar curvature τ is given by

$$(3.4) \quad \tau = \sum_{l=1}^n S(e_l, e_l) = \frac{c}{4} \{n(n-1) + 12n \cos^2 \theta\} + n^2 \|H\|^2 - \|h\|^2.$$

We put

$$(3.5) \quad \epsilon = \tau - \frac{n^2}{2} \|H\|^2 - \frac{c}{4} \{n(n-1) + 12n \cos^2 \theta\}.$$

Then from equations (3.4) and (3.5), we get

$$(3.6) \quad n^2 \|H\|^2 = 2(\epsilon + \|h\|^2).$$

With respect to above orthonormal basis, the equation (3.6) takes the form

$$(3.7) \quad \left(\sum_{i=1}^n h_{ii}^{n+1} \right)^2 = 2 \left\{ \epsilon + \sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right\}.$$

If we set $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n-1} h_{ii}^{n+1}$ and $a_3 = h_{nn}^{n+1}$, then (3.7) becomes

$$(3.8) \quad \left(\sum_{i=1}^3 a_i \right)^2 = 2 \left\{ \epsilon + \sum_{i=1}^3 a_i^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 - \sum_{2 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} \right\}.$$

Thus a_1, a_2, a_3 satisfy the Lemma 2.1 of Chen for $(n = 3)$, i.e.,

$$\left(\sum_{i=1}^3 a_i \right)^2 = 2 \left(b + \sum_{i=1}^3 a_i^2 \right).$$

So, we have $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this implies that equation (3.8) becomes

$$(3.9) \quad \sum_{1 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} \geq \epsilon + 2 \sum_{i < j} (h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2,$$

or equivalently

$$(3.10) \quad \frac{n^2}{2} \|H\|^2 + \frac{c}{4} [n(n-1) + 12n \cos^2 \theta] \geq \tau - \sum_{1 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2 \sum_{i < j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2.$$

Using again the equation of Gauss, we have

$$(3.11) \quad \begin{aligned} & \tau - \sum_{1 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2 \sum_{i < j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 \\ &= 2S(e_n, e_n) + \frac{c}{4} [(n-1)(n-2) + 12(n-1) \cos^2 \theta] \\ & \quad + 2 \sum_{i < n} (h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m} \{ (h_{nn}^r)^2 + 2 \sum_{i=1}^{n-1} (h_{in}^r)^2 + (\sum_{j=1}^{n-1} h_{jj}^r)^2 \}, \end{aligned}$$

where S is the Ricci tensor of M .

Combining (3.10) and (3.11), we obtain

$$(3.12) \quad \frac{n^2}{2} \|H\|^2 + \frac{c}{4} [2(n-1) + 12 \cos^2 \theta] \\ \geq 2S(e_n, e_n) + 2 \sum_{i < n} (h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m} \left\{ \sum_{i=1}^n (h_{in}^r)^2 + \left(\sum_{j=1}^{n-1} h_{jj}^r \right)^2 \right\}.$$

Thus, we have

$$\text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (n-1)c + 6c \cos^2 \theta\},$$

which proves (3.1).

(II) Assume $H(p) = 0$. Equality holds in (3.1) if and only if

$$(3.13) \quad h_{1n}^r = \dots = h_{n-1,n}^r = 0, \quad h_{nn}^r = \sum_{i=1}^{n-1} h_{ii}^r, \quad r \in \{n+1, \dots, 4m\}.$$

Then $h_{in}^r = 0, \forall i \in \{1, \dots, n\}, r \in \{n+1, \dots, 4m\}$, i.e. X belongs to the relative null space N_p . $\square$

Theorem 3.2. *Let M be an n -dimensional quaternion bi-slant submanifold of a $4m$ -dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c . Then*

(I) *For each unit vector $X \in T_p M$, if*

(a) *X is tangent to D_1 , we have*

$$(3.14) \quad \text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (n-1)c + 6c \cos^2 \theta_1\}$$

and

(b) *X is tangent to D_2 , we have*

$$(3.15) \quad \text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (n-1)c + 6c \cos^2 \theta_2\}$$

(II) *If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.14) and (3.15) if and only if X belongs to the relative null space N_p .*

Proof. Let $p \in M$, we choose an orthonormal basis $\{e_1, \dots, e_n\}$ for $T_p M$ and $\{e_{n+1}, \dots, e_{4m}\}$ for the normal space $T_p^\perp M$ at p such that $e_n = X$ and e_{n+1} is parallel to the mean curvature vector $\bar{H}(p)$.

From the equation of Gauss, the scalar curvature τ is given by

$$(3.16) \quad \tau = \sum_{l=1}^n S(e_l, e_l) \\ = \frac{c}{4} \{n(n-1) + 12(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)\} + n^2 \|H\|^2 - \|h\|^2.$$

We put

$$(3.17) \quad \epsilon = \tau - \frac{n^2}{2} \|H\|^2 - \frac{c}{4} \{n(n-1) + 12(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)\}.$$

Then from equations (3.16) and (3.17), we get

$$(3.18) \quad n^2 \|H\|^2 = 2(\epsilon + \|h\|^2).$$

With respect to above orthonormal basis, the equation (3.18) takes the form

$$(3.19) \quad \left(\sum_{i=1}^n h_{ii}^{n+1} \right)^2 = 2 \left\{ \epsilon + \sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right\}.$$

If we set $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n-1} h_{ii}^{n+1}$ and $a_3 = h_{nn}^{n+1}$, then (3.19) becomes

$$(3.20) \quad \left(\sum_{i=1}^3 a_i \right)^2 = 2 \left\{ \epsilon + \sum_{i=1}^3 a_i^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 - \sum_{2 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} \right\}.$$

Thus a_1, a_2, a_3 satisfy the Lemma 2.1 of Chen for $(n = 3)$, i.e.,

$$\left(\sum_{i=1}^3 a_i \right)^2 = 2 \left(b + \sum_{i=1}^3 a_i^2 \right).$$

So, we have $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this implies that equation (3.20) becomes

$$(3.21) \quad \sum_{1 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} \geq \epsilon + 2 \sum_{i < j} (h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2,$$

or equivalently

$$(3.22) \quad \begin{aligned} & \frac{n^2}{2} \|H\|^2 + \frac{c}{4} [n(n-1) + 12(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)] \\ & \geq \tau - \sum_{1 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2 \sum_{i < j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2. \end{aligned}$$

Now, we consider two cases:

(a) If X is tangent to D_1 , we have

$$\begin{aligned}
 (3.23) \quad \tau - \sum_{1 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2 \sum_{i < j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 \\
 = 2S(e_n, e_n) + \frac{c}{4}[(n-1)(n-2) + 12\{(d_1-1)\cos^2 \theta_1 + d_2 \cos^2 \theta_2\}] \\
 + 2 \sum_{i < n} (h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m} \left\{ (h_{nn}^r)^2 + 2 \sum_{i=1}^{n-1} (h_{in}^r)^2 + \left(\sum_{j=1}^{n-1} h_{jj}^r \right)^2 \right\},
 \end{aligned}$$

where S is the Ricci tensor of M .

Combining (3.22) and (3.23), we obtain

$$\begin{aligned}
 (3.24) \quad \frac{n^2}{2} \|H\|^2 + \frac{c}{4}[2(n-1) + 12 \cos^2 \theta_1] \\
 \geq 2S(e_n, e_n) + 2 \sum_{i < n} (h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m} \left\{ \sum_{i=1}^n (h_{in}^r)^2 + \left(\sum_{j=1}^{n-1} h_{jj}^r \right)^2 \right\}.
 \end{aligned}$$

Thus, we have

$$\text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (n-1)c + 6c \cos^2 \theta_1\},$$

which proves (3.14).

(b) If X is tangent to D_2 , we have

$$\begin{aligned}
 (3.25) \quad \tau - \sum_{1 \leq \alpha \neq \beta \leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2 \sum_{i < j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 \\
 = 2S(e_n, e_n) + \frac{c}{4}[(n-1)(n-2) + 12\{d_1 \cos^2 \theta_1 + (d_2-1) \cos^2 \theta_2\}] \\
 + 2 \sum_{i < n} (h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m} \left\{ (h_{nn}^r)^2 + 2 \sum_{i=1}^{n-1} (h_{in}^r)^2 + \left(\sum_{j=1}^{n-1} h_{jj}^r \right)^2 \right\},
 \end{aligned}$$

where S is the Ricci tensor of M .

Combining (3.22) and (3.25), we obtain

$$\begin{aligned}
 (3.26) \quad \frac{n^2}{2} \|H\|^2 + \frac{c}{4}[2(n-1) + 12 \cos^2 \theta_2] \\
 \geq 2S(e_n, e_n) + 2 \sum_{i < n} (h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m} \left\{ \sum_{i=1}^n (h_{in}^r)^2 + \left(\sum_{j=1}^{n-1} h_{jj}^r \right)^2 \right\}.
 \end{aligned}$$

Thus, we have

$$\text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (n-1)c + 6c \cos^2 \theta_2\},$$

which proves (3.15).

(II) Assume $H(p) = 0$. Equality holds in (3.14) and (3.15) if and only if

$$(3.27) \quad h_{1n}^r = \dots = h_{n-1,n}^r = 0, \quad h_{nn}^r = \sum_{i=1}^{n-1} h_{ii}^r, \quad r \in \{n+1, \dots, 4m\}.$$

Then $h_{in}^r = 0, \forall i \in \{1, \dots, n\}, r \in \{n+1, \dots, 4m\}$, i.e. X belongs to the relative null space N_p . $\square$

Now, we can state the following:

Corollary 3.3. *Let M be an n -dimensional quaternion semi-slant submanifold of a $4m$ -dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c . Then*

(I) *For each unit vector $X \in T_p M$, if*

(a) *X is tangent to D_1 , we have*

$$(3.28) \quad \text{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\}$$

and

(b) *X is tangent to D_2 , we have*

$$(3.29) \quad \text{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c \cos^2 \theta\}.$$

(II) *If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.28) and (3.29) if and only if X belongs to the relative null space N_p .*

Corollary 3.4. *Let M be an n -dimensional quaternion submanifold of a $4m$ -dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c . Then*

(I) *For each unit vector $X \in T_p M$, we have*

$$(3.30) \quad \text{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\}.$$

(II) *If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.30) if and only if X belongs to the relative null space N_p .*

Corollary 3.5. *Let M be an n -dimensional totally real submanifold of a $4m$ -dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c . Then*

(I) *For each unit vector $X \in T_p M$, we have*

$$(3.31) \quad \text{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c\}.$$

(II) *If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.31) if and only if X belongs to the relative null space N_p .*

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