

CONTINUITY FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR ON BESOV SPACES

TAN LU, CHUANGXIA HUANG AND LIU LANZHE

ABSTRACT. In this paper, we prove the continuity for the multilinear commutator associated to the Littlewood-Paley operator on the Besov spaces.

1. INTRODUCTION

As the development of the singular integral operators, their commutators have been well studied (see [1, 2, 3, 14]). From [2, 3, 9, 13], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce the multilinear commutator associated to the Littlewood-Paley operator and prove the continuity properties for the multilinear commutator on the Besov spaces.

2. PRELIMINARIES AND THEOREM

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f , the sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [15, 16])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

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For $\beta \geq 0$, the Besov space $\dot{\Lambda}_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left| \Delta_h^{[\beta]+1} f(x) \right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k -th difference operator (see [13]).

For $b_j \in \dot{\Lambda}_\beta(R^n)$ ($j = 1, \dots, m$), set

$$\|\vec{b}\|_{\dot{\Lambda}_\beta} = \prod_{j=1}^m \|b_j\|_{\dot{\Lambda}_\beta}.$$

Given some functions b_j ($j = 1, \dots, m$) and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{\dot{\Lambda}_\beta} = \|b_{\sigma(1)}\|_{\dot{\Lambda}_\beta} \cdots \|b_{\sigma(j)}\|_{\dot{\Lambda}_\beta}$.

Definition 1. Let $0 < p, q \leq \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0 .

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha, p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha, p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p \right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha, p}(R^n) = \{f \in L_{loc}^q(R^n) : \|f\|_{K_q^{\alpha, p}} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha, p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p + \|f\chi_{B_0}\|_{L^q}^p \right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 2. Let $1 \leq q < \infty$, $\alpha \in R$. The central Campanato space is defined by (see [17])

$$CL_{\alpha, q}(R^n) = \{f \in L_{loc}^q(R^n) : \|f\|_{CL_{\alpha, q}} < \infty\},$$

where

$$\|f\|_{CL_{\alpha, q}} = \sup_{r>0} |B(0, r)|^{-\alpha} \left(\frac{1}{|B(0, r)|} \int_{B(0, r)} |f(x) - f_{B(0, r)}|^q dx \right)^{1/q}.$$

Definition 3. Fix $\delta > 0$. Let ψ be a fixed function which satisfies the following properties:

- (1) $\int_{R^n} \psi(x) dx = 0$;
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$;
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\epsilon(1 + |x|)^{-(n+\epsilon-\delta)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear commutator is defined by

$$S_\delta^{\vec{b}}(f)(x) = \left[\int \int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x, y) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(z)) \psi_t(y - z) f(z) dz,$$

and $\psi_t(x) = t^{-n+\delta} \psi(x/t)$ for $t > 0$. We also define that

$$S_\delta(f)(x) = \left(\int \int_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [16]).

Let H be the space $H = h : \|h\| = (\int \int_{R_+^{n+1}} |h(y, t)|^2 \frac{dy dt}{t^{n+1}})^{1/2} < \infty$, then, for each fixed $x \in R_n$, $F_t^{\vec{b}}(f)(x, y)$ may be viewed as a mapping from $[0, +\infty)$ to H , and it is clear that

$$S_\delta(f)(x) = \|\chi_{\Gamma(x)} F_t(f)(x)\|$$

and

$$S_\delta^{\vec{b}}(f)(x) = \|\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(x, y)\|.$$

Note that when $b_1 = \dots = b_m$, $S_\delta^{\vec{b}}$ is just the commutator of order m . It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [2, 3, 4, 7, 6, 5, 8, 9, 10, 13, 14]). Our main purpose is to study the boundedness properties for the multilinear commutator on Besov spaces.

Now, we state our theorems as following.

Theorem 1. Let $0 < \delta < n$, $1 < r < n/\delta$, $1/s = 1/r - \delta/n$, $0 < \beta < 1/2m$ and $b_j \in \dot{\Lambda}_\beta(R^n)$ for $j = 1, \dots, m$. Then $S_\delta^{\vec{b}}$ is bounded from $L^p(R^n)$ to $\dot{\Lambda}_{(\delta+m\beta)-n/p}(R^n)$ for any $n/(\delta + m\beta) \leq p \leq n/\delta$.

Theorem 2. Let $0 < \delta < n$, $0 < \beta < 1/2m$, $1 < q_1 < n/(\delta + m\beta)$, $1/q_2 = 1/q_1 - (\delta + m\beta)/n$, $-n/q_2 - 1/2 < \alpha \leq -n/q_2$ and $b_j \in \dot{\Lambda}_\beta(R^n)$ for $j = 1, \dots, m$. Then $S_\delta^{\vec{b}}$ is bounded from $\dot{K}_{q_1}^{\alpha, \infty}(R^n)$ to $CL_{-\alpha/n-1/q_2, q_2}(R^n)$.

Remark 1. Theorem 2 also hold for the nonhomogeneous Herz type Hardy space.

3. PROOFS OF THEOREMS

To prove the theorems, we need the following lemmas.

Lemma 1 (see [13]). *For $0 < \beta < 1, 1 \leq p \leq \infty$, we have*

$$\begin{aligned} \|b\|_{\dot{\Lambda}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p} \\ &\approx \sup_Q \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - c| dx \\ &\approx \sup_Q \inf_c \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - c|^p dx \right)^{1/p}. \end{aligned}$$

Lemma 2 (see [12]). *For $\alpha < 0, 0 < q < \infty$, we have*

$$\|f\|_{\dot{K}_q^{\alpha, \infty}} \approx \sup_{\mu \in \mathbb{Z}} 2^{\mu\alpha} \|f \chi_{B_\mu}\|_{L^q}.$$

Lemma 3. *Let $0 < \eta < n, 1 < p < n/\eta$. Suppose $b \in \dot{\Lambda}_\beta(R^n)$, then*

$$|b_{2^{k+1}B} - b_B| \leq C \|b\|_{\dot{\Lambda}_\beta} k |2^{k+1}B|^{\beta/n} \text{ for } k \geq 1.$$

Proof.

$$\begin{aligned} |b_{2^{k+1}B} - b_B| &\leq \sum_{j=0}^k |b_{2^{j+1}B} - b_{2^jB}| \\ &\leq \sum_{j=0}^k \frac{1}{|2^jB|} \int_{2^jB} |b(y) - b_{2^{j+1}B}| dy \\ &\leq C \sum_{j=0}^k \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^p dy \right)^{1/p} \\ &\leq C \|b\|_{\dot{\Lambda}_\beta} \sum_{j=0}^k |2^{j+1}B|^{\beta/n} \leq C \|b\|_{\dot{\Lambda}_\beta} k |2^{k+1}B|^{\beta/n}. \end{aligned}$$

□

Lemma 4 (see [16]). *Let $0 < \delta < n, 1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then S_δ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Lemma 5 (see [7]). *Let $0 \leq \eta < n, 1 < r < n/\eta, 1/r - 1/s = \eta/n$ and $b_j \in \dot{\Lambda}_\beta(R^n)$ for $j = 1, \dots, m$. Then $S_\delta^{\vec{b}}$ is bounded from $L^r(R^n)$ to $L^s(R^n)$.*

Proof of Theorem 1. It is only to prove that there exists a constant C_0 such that

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |S_\delta^{\vec{b}}(f)(x) - C_0| dx \leq C \|f\|_{L^p}.$$

Fix a cube Q , $Q = Q(x_0, d)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_Q$, $f_2 = f\chi_{(R^n \setminus Q)}$.

When $m = 1$, for $C_0 = S_\delta((b_1)_Q - b_1)f_2(x_0)$, we have

$$S_\delta^{b_1}(f)(x) = (b_1(x) - (b_1)_Q)S_\delta(f)(x) - S_\delta((b_1 - (b_1)_Q)f_1)(x) - S_\delta((b_1 - (b_1)_Q)f_2)(x).$$

Then

$$\begin{aligned} & |S_\delta^{b_1}(f)(x) - S_\delta((b_1)_Q - b_1)f_2(x_0)| = \\ & = ||\chi_{\Gamma(x)}F_t^{b_1}(f)(x, y)| - |\chi_{\Gamma(x_0)}F_t((b_1)_Q - b_1)f_2(y)|| \\ & \leq ||\chi_{\Gamma(x)}F_t^{b_1}(f)(x, y) - \chi_{\Gamma(x_0)}F_t((b_1)_Q - b_1)f_2(y)|| \\ & \leq ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_Q)F_t(f)(y)|| + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_Q)f_1)(y)|| \\ & \quad + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_Q)f_2)(y) - \chi_{\Gamma(x_0)}F_t((b_1 - (b_1)_Q)f_2)(y)|| \\ & = A(x) + B(x) + C(x). \end{aligned}$$

For $A(x)$, for $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, by the boundness of S_δ from $L^p(R^n)$ to $L^q(R^n)$, by Hölder's inequality with exponent $1/q + 1/q' = 1$ and Lemma 1, we have

$$\begin{aligned} & \frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |A(x)| dx \leq \\ & \leq C \frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \left(\int_Q |(b_1(x) - (b_1)_Q)|^{q'} dx \right)^{1/q'} \left(\int_Q |S_\delta(f)(x)|^q dx \right)^{1/q} \\ & \leq C \frac{|Q|^{\beta/n+1/q'}}{|Q|^{1+(\delta+\beta)/n-1/p}} \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b_1(x) - (b_1)_Q)|^{q'} dx \right)^{1/q'} \left(\int_Q |f(x)|^p dx \right)^{1/p} \\ & \leq C \frac{|Q|^{1+(\delta+\beta)/n-1/p}}{|Q|^{1+(\delta+\beta)/n-1/p}} \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p} \\ & \leq C \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p}. \end{aligned}$$

For $B(x)$, denoting $p = rt$, $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of S_δ from $L^r(R^n)$ to $L^s(R^n)$, by Hölder's inequality with exponent $1/t + 1/t' = 1$ and Lemma 1, we have

$$\begin{aligned} & \frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |B(x)| dx \leq \\ & \leq C \frac{1}{|Q|^{(\delta+\beta)/n-1/p}} \left(\frac{1}{|Q|} \int_{R^n} |S_\delta(b_1(x) - (b_1)_Q)f\chi_Q(x)|^s dx \right)^{1/s} \\ & \leq C \frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \left(\int_Q |(b_1(x) - (b_1)_Q)f(x)|^r dx \right)^{1/r} \\ & \leq C \frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \left(\int_Q |b_1(x) - (b_1)_Q|^{rt'} dx \right)^{1/rt'} \left(\int_Q |f(x)|^{rt} dx \right)^{1/rt} \end{aligned}$$

$$\begin{aligned}
&\leq C \frac{|Q|^{\beta/n+1/rt'}}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_Q|^{rt'} dx \right)^{1/rt'} \left(\int_Q |f(x)|^{rt} dx \right)^{1/rt} \\
&\leq C \frac{|Q|^{(\delta+\beta)/n-1/p+1/s}}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p} \\
&\leq C \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

For $C(x)$, by the Minkowski's inequality, we have

$$\begin{aligned}
C(x) &\leq \left[\int \int_{R_+^{n+1}} \left(\int_{Q^c} |\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}| |b_1(z) - (b_1)_Q| |\psi_t(y-z)| |f(z)| \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
&\leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| \\
&\quad \times \left| \int \int_{|x-y|\leq t} \frac{t^{1-n} dydt}{(t+|y-z|)^{2n+2-2\delta}} - \int \int_{|x_0-y|\leq t} \frac{t^{1-n} dydt}{(t+|y-z|)^{2n+2-2\delta}} \right|^{1/2} dz \\
&\leq \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| \times \\
&\quad \times \left(\int \int_{|y|\leq t, |x+y-z|\leq t} \left| \frac{1}{(t+|x+y-z|)^{2n+2-2\delta}} - \frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}} \right| \frac{dydt}{t^{n-1}} \right)^{1/2} dz \\
&\leq \int_{Q^c} |b_1(z) - (b_1)_{2Q}| |f(z)| \left(\int \int_{|y|\leq t, |x+y-z|\leq t} \frac{|x-x_0| t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}} dydt \right)^{1/2} dz,
\end{aligned}$$

by $2t + |x+y-z| \geq 2t + |x-z| - |y| \geq t + |x-z|$, for $|y| \leq t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+3-2\delta}} = C|x-z|^{-2n-1+2\delta}.$$

Thus, for $x \in Q$, by Hölder's inequality with exponent $1/p + 1/p' = 1$ and Lemma 1, 3, we have

$$\begin{aligned}
C(x) &\leq \\
&\leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| \left(\int \int_{|y|\leq t} \frac{2^{2n+3} |x_0-x| t^{1-n} dydt}{(2t+2|x+y-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
&\leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| |x-x_0|^{1/2} \left(\int \int_{|y|\leq t} \frac{t^{1-n} dydt}{(2t+|x+y-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
&\leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| |x-x_0|^{1/2} \left(\int \int_{|y|\leq t} \frac{t^{1-n} dydt}{(t+|x-z|)^{2n+3-2\delta}} \right)^{1/2} dz
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| |x - x_0|^{1/2} \left(\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
&\leq C \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^kQ} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-2\delta)} |b_1(z) - (b_1)_Q| |f(z)| dz \\
&\leq C \sum_{k=0}^\infty 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p} \times \\
&\quad \times \left(\int_{2^{k+1}Q} |b_1(z) - (b_1)_{2Q}|^{p'} dz \right)^{1/p'} \\
&\leq C \sum_{k=0}^\infty 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p} \times \\
&\quad \times \left[\int_{2^{k+1}Q} (|b_1(z) - (b_1)_{2^{k+1}Q}|^{p'} + |(b_1)_{2^{k+1}Q} - (b_1)_{2Q}|^{p'}) dz \right]^{1/p'} \\
&\leq C \sum_{k=0}^\infty 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p} \times \\
&\quad \times \left[|2^{k+1}Q|^{\beta/n+1/p'} \frac{1}{|2^{k+1}Q|^{\beta/n}} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b_1(z) - (b_1)_{2^{k+1}Q})|^{p'} dz \right)^{1/p'} \right. \\
&\quad \left. + |(b_1)_{2^{k+1}Q} - (b_1)_Q| |2^{k+1}Q|^{1/p'} \right] \\
&\leq C \sum_{k=0}^\infty 2^{-k/2} \frac{|2^{k+1}Q|^{\beta/n+1/p'} \|b_1\|_{\dot{\lambda}_\beta} + k |2^{k+1}Q|^{\beta/n+1/p'} \|b_1\|_{\dot{\lambda}_\beta}}{|2^{k+1}Q|^{1-\delta/n}} \|f\|_{L^p} \\
&\leq C \sum_{k=0}^\infty k 2^{-k/2} |2^{k+1}Q|^{\beta/n+1/p'-1+\delta/n} \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p} \\
&\leq C \sum_{k=0}^\infty k 2^{k(-1/2+\delta+\beta-n/p)} |Q|^{\beta/n+1/p'-1+\delta/n} \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p} \\
&\leq C |Q|^{(\delta+\beta)/n-1/p} \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |C(x)| dx &\leq C \frac{\|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p}}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |Q|^{(\delta+\beta)/n-1/p} dx \\
&\leq C \|b_1\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

This completes the case $m = 1$.

Now, we consider the *Case* $m \geq 2$. we have, for $b = (b_1, \dots, b_m)$,

$$F_t^{\vec{b}}(f)(x, y) =$$

$$\begin{aligned}
&= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_Q) - (b_j(z) - (b_j)_Q)] \psi_t(y - z) f(z) dz \\
&= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_Q)_\sigma \int_{R^n} (b(z) - (b)_Q)_{\sigma^c} \psi_t(y - z) f(z) dz \\
&= \prod_{j=1}^m (b_j(x) - (b_j)_Q) F_t(f)(y) + (-1)^m F_t\left(\prod_{j=1}^m (b_j - (b_j)_Q)\right) f(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_Q)_\sigma \int_{R^n} (b(z) - b(x))_{\sigma^c} \psi_t(y - z) f(z) dz \\
&= \prod_{j=1}^m (b_j(x) - (b_j)_Q) F_t(f)(y) + (-1)^m F_t\left(\prod_{j=1}^m (b_j - (b_j)_Q)\right) f(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (b(x) - (b)_Q)_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x, y).
\end{aligned}$$

Thus, set $C_0 = S_\delta\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2\right)(x_0)$,

$$\begin{aligned}
&|S_\delta^{\vec{b}}(f)(x) - S_\delta\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2\right)(x_0)| \leq \\
&\leq \|\chi_{\Gamma(x)} \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) F_t(f)(x)\| \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\chi_{\Gamma(x)} (b(x) - (b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)\| \\
&\quad + \|\chi_{\Gamma(x)} F_t\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_1\right)(x)\| \\
&\quad + \|\chi_{\Gamma(x)} F_t\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2\right)(x) - \chi_{\Gamma(x_0)} F_t\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2\right)(x_0)\| \\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

For $I_1(x)$, for $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, by the boundness of S_δ from $L^p(R^n)$ to $L^q(R^n)$, by Hölder's inequality with exponent $1/q'_1 + \dots + 1/q'_m + 1/q = 1$ and Lemma 1, we have

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |I_1(x)| dx =$$

$$\begin{aligned}
&= \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_Q) S_\delta(f)(x) \right| dx \\
&\leq C \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \prod_{j=1}^m \left(\int_Q |b_j(x) - (b_j)_Q|^{q'_j} dx \right)^{1/q'_j} \left(\int_Q |S_\delta(f)(x)|^q dx \right)^{1/q} \\
&\leq C \frac{|Q|^{m\beta/n+1/q'_1+\dots+q'_m}}{|Q|^{1+(\delta+m\beta)/n-1/p}} \prod_{j=1}^m \frac{1}{|Q|^{m\beta/n}} \left(\frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_Q|^{q'_j} dx \right)^{1/q'_j} \times \\
&\quad \times \left(\int_Q |f(x)|^p dx \right)^{1/p} \\
&\leq C \frac{|Q|^{1+(\delta+m\beta)/n-1/p}}{|Q|^{1+(\delta+m\beta)/n-1/p}} \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} \\
&\leq C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

For $I_2(x)$, denoting $p = rt$, $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of S_δ from $L^r(R^n)$ to $L^s(R^n)$, by Hölder's inequality with $1/s' + 1/s = 1$, $1/t' + 1/t = 1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |I_2(x)| dx = \\
&= \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |(b(x) - b_Q)_\sigma S_\delta((b - b_Q)_{\sigma^c} f)(x)| dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \left(\int_Q |(b(x) - b_Q)_\sigma|^{s'} dx \right)^{1/s'} \times \\
&\quad \times \left(\int_Q |S_\delta((b - b_Q)_{\sigma^c} f)(x)|^s dx \right)^{1/s} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{|Q|^{|\sigma|\beta/n+1/s'}}{|Q|^{1+(\delta+m\beta)/n-1/p}} \frac{1}{|Q|^{|\sigma|\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b(x) - b_Q)_\sigma|^{s'} dx \right)^{1/s'} \\
&\quad \times \left(\int_Q |(b - b_Q)_{\sigma^c} f(x)|^r dx \right)^{1/r} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{|Q|^{|\sigma|\beta/n+1/s'}}{|Q|^{1+m\beta/n-1/p}} \frac{1}{|Q|^{|\sigma|\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b(x) - b_Q)_\sigma|^{s'} dx \right)^{1/s'} \\
&\quad \times \left(\int_Q |(b(x) - b_Q)_{\sigma^c}|^{rt'} dx \right)^{1/rt'} \left(\int_Q |f(x)|^{rt} dx \right)^{1/rt}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{|Q|^{\sigma|\beta/n+1/s'|}}{|Q|^{1+(\delta+m\beta)/n-1/p}} \frac{1}{|Q|^{\sigma|\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b(x) - b_Q)_\sigma|^{s'} dx \right)^{1/s'} \\
&\quad \times |Q|^{\sigma^c|\beta/n+1/rt'|} \frac{1}{|Q|^{\sigma^c|\beta/n}} \left(\frac{1}{|Q|} \int_Q |(b(x) - b_Q)_{\sigma^c}|^{rt'} dx \right)^{1/rt'} \|f\|_{L^p} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{|Q|^{m\beta/n+1/s'+1/rt'}}{|Q|^{1+(\delta+m\beta)/n-1/p}} \|\vec{b}_\sigma\|_{\dot{\lambda}_\beta} \|\vec{b}_{\sigma^c}\|_{\dot{\lambda}_\beta} \|f\|_{L^p} \\
&\leq C \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

For $I_3(x)$, for $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of S_δ from $L^r(R^n)$ to $L^s(R^n)$, taking $1 < r < p < \infty$, $p = rt$, $1/t_1 + \dots + 1/t_m + 1/t = 1$, by Hölder's inequality and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |I_3(x)| dx \\
&\leq \frac{1}{|Q|^{(\delta+m\beta)/n-1/p}} \left(\frac{1}{|Q|} \int_{R^n} |S_\delta(\prod_{j=1}^m (b_j - (b_j)_Q)) f \chi_B(x)|^s dx \right)^{1/s} \\
&\leq \frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}} \left(\int_Q |(\prod_{j=1}^m (b_j - (b_j)_Q) f(x)|^r dx \right)^{1/r} \\
&\leq \frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}} \prod_{j=1}^m \left(\int_Q |b_j(x) - (b_j)_Q|^{rt_j} dx \right)^{1/rt_j} \left(\int_Q |f(x)|^{rt} dx \right)^{1/rt} \\
&\leq \frac{|Q|^{m\beta/n+1/rt_1+\dots+1/rt_m}}{|Q|^{(\delta+m\beta)/n-1/p+1/s}} \prod_{j=1}^m \frac{1}{|Q|^{m\beta/n}} \left(\frac{1}{|Q|} \int_Q |b_j(x) - (b_j)_Q|^{rt_j} dx \right)^{1/rt_j} \|f\|_{L^p} \\
&\leq C \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

For $I_4(x)$, similar to the proof of $C(x)$ in the case $m = 1$, we get

$$I_4(x) \leq C \int_{(2Q)^c} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_Q) \right| |f(z)| dz.$$

Thus, we choose $1 < p_j < \infty$, $j = 1, \dots, m$, $1/p_1 + \dots + 1/p_m + 1/p = 1$, by Hölder's inequality and Lemma 1, 3, we have

$$\begin{aligned}
I_4(x) &\leq \\
&\leq C \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_Q) \right| |f(z)| dz \\
&\leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p} \times
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{j=1}^m \left(\int_{2^{k+1}Q} |(b_j(z) - (b_j)_Q)|^{p'_j} dz \right)^{1/p'_j} \\
& \leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p} \\
& \times \prod_{j=1}^m \left[\int_{2^{k+1}Q} (|b_j(z) - (b_j)_{2^{k+1}Q}|^{p'_j} + |(b_j)_{2^{k+1}Q} - (b_j)_{2Q}|^{p'_j}) dz \right]^{1/p'_j} \\
& \leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \left(\int_{2^{k+1}Q} |f(z)|^p dz \right)^{1/p} \\
& \times \prod_{j=1}^m \left[|2^{k+1}Q|^{m\beta/n+1/p'} \frac{1}{|2^{k+1}Q|^{m\beta/n}} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b_j(z) - (b_j)_{2^{k+1}Q})|^{p'_j} dz \right)^{1/p'} \right. \\
& \quad \left. + |(b_j)_{2^{k+1}Q} - (b_j)_Q| |2^{k+1}Q|^{1/p'} \right] \\
& \leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \times \\
& \times \prod_{j=1}^m \left[|2^{k+1}Q|^{m\beta/n+1/p'} \|b_j\|_{\dot{\lambda}_\beta} + k |2^{k+1}Q|^{m\beta/n+1/p'} \|b_j\|_{\dot{\lambda}_\beta} \right] \|f\|_{L^p} \\
& \leq C \sum_{k=0}^{\infty} k 2^{k(-1/2+\delta+m\beta-n/p)} |Q|^{m\beta/n+1/p'-1+\delta/n} \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{L^p} \\
& \leq C |Q|^{(\delta+m\beta)/n-1/p} \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |I_4(x)| dx \leq \\
& \leq C \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{L^p} \frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |Q|^{(\delta+m\beta)/n-1/p} dx \leq C \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

□

Proof of Theorem 2. Fix a ball $B = B(0, l)$, there exists $\epsilon_0 \in \mathbf{Z}$ such that $2^{\epsilon_0-1} \leq l < 2^{\epsilon_0}$. We choose x_0 such that $2l < |x_0| < 3l$. It is only to prove that

$$2^{\epsilon_0(\alpha+n/q_2)} \left(\frac{1}{2^{\epsilon_0 n}} \int_{|x| < 2^{\epsilon_0}} |S_\delta^\vec{b}(f)(x) - S_\delta^\vec{b}(f_2)(x_0)|^{q_2} dx \right)^{1/q_2} \leq C \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}.$$

We write, for $f_1 = f\chi_{4B_{\epsilon_0}}$ and $f_2 = f\chi_{R^n \setminus 4B_{\epsilon_0}}$, then

$$|S_\delta^\vec{b}(f)(x) - S_\delta^\vec{b}(f_2)(x_0)| \leq |S_\delta^\vec{b}(f_1)(x)| + |S_\delta^\vec{b}(f_2)(x) - S_\delta^\vec{b}(f_2)(x_0)|.$$

So

$$\begin{aligned}
& 2^{\epsilon_0(\alpha+n/q_2)} \left(\frac{1}{2^{\epsilon_0 n}} \int_{|x|<2^{\epsilon_0}} |S_{\delta}^{\vec{b}}(f)(x) - S_{\delta}^{\vec{b}}(f_2)(x_0)|^{q_2} dx \right)^{1/q_2} \\
& \leq 2^{\epsilon_0(\alpha+n/q_2)} \left(\frac{1}{2^{\epsilon_0 n}} \int_{|x|<2^{\epsilon_0}} |S_{\delta}^{\vec{b}}(f_1)(x)|^{q_2} dx \right)^{1/q_2} \\
& \quad + 2^{\epsilon_0(\alpha+n/q_2)} \left(\frac{1}{2^{\epsilon_0 n}} \int_{|x|<2^{\epsilon_0}} |S_{\delta}^{\vec{b}}(f_2)(x) - S_{\delta}^{\vec{b}}(f_2)(x_0)|^{q_2} dx \right)^{1/q_2} \\
& = J_1 + J_2.
\end{aligned}$$

For J_1 , by the (L^{q_1}, L^{q_2}) -boundedness of $S_{\delta}^{\vec{b}}$ (see Lemma 4) and Lemma 2, we get

$$\begin{aligned}
J_1 & \leq C 2^{\epsilon_0(\alpha+n/q_2)} 2^{-\epsilon_0 n/q_2} \left(\int_{R^n} |f_1(x)|^{q_1} dx \right)^{1/q_1} \\
& \leq C 2^{\epsilon_0 \alpha} \|f \chi_{B_{\epsilon_0}}\|_{L^{q_1}} \\
& \leq C \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}.
\end{aligned}$$

For J_2 , similar to the estimates of Theorem 1, set $1/v_1 + \dots + 1/v_m + 1/q_1 = 1$, by Hölder's inequality and recall that $-1/q_2 < \alpha$, $1/q_2 = 1/q_1 - (\delta + m\beta)/n$, we obtain

$$\begin{aligned}
& |S_{\delta}^{\vec{b}}(f_2)(x) - S_{\delta}^{\vec{b}}(f_2)(x_0)| \leq \\
& \leq |S_{\delta}(\prod_{j=1}^m (b_j - (b_j)_B))(f_2)(x) - S_{\delta}(\prod_{j=1}^m (b_j - (b_j)_B))(f_2)(x_0)| \\
& \quad + |\prod_{j=1}^m (b_j(x) - (b_j)_B)| |S_{\delta}(f_2)(x) - S_{\delta}(f_2)(x_0)| = W_1(x) + W_2(x).
\end{aligned}$$

For $W_1(x)$, similar to the proof of $C(x)$ in Theorem 1, by the Minkowski's inequality, we have

$$\begin{aligned}
W_1(x) & \leq C \sum_{k=1}^{\infty} \int_{B_{\epsilon_0+k}} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_B) \right| |f(z)| dz \\
& \leq C \sum_{k=1}^{\infty} \int_{B_{\epsilon_0+k}} \frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}} \left| \prod_{j=1}^m (b_j(z) - (b_j)_B) \right| |f(z)| dz \\
& \leq C \sum_{k=1}^{\infty} \frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}} \prod_{j=1}^m \left(\int_{B_{\epsilon_0+k}} |b_j(z) - (b_j)_B|^{v_j} dy \right)^{1/v_j} \times \\
& \quad \times \left(\int_{B_{\epsilon_0+k}} |f(z)|^{q_1} dz \right)^{1/q_1}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=1}^{\infty} \frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}} 2^{(\epsilon_0+k)(m\beta+n/v_1+\dots+n/v_m)} \\
&\times \prod_{j=1}^m \frac{1}{|B_{\epsilon_0+k}|^{m\beta/n}} \left(\frac{1}{|B_{\epsilon_0+k}|} \int_{B_{\epsilon_0+k}} |b_j(z) - (b_j)_B|^{v_j} dz \right)^{1/v_j} \|f\chi_{\epsilon_0+k}\|_{L^{q_1}} \\
&\leq C \sum_{k=1}^{\infty} 2^{k(m\beta+n-n/q_1-n+\delta-1/2)} 2^{\epsilon_0(m\beta+n-n/q_1-n+\delta)} \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\chi_{\epsilon_0+k}\|_{L^{q_1}} \\
&\leq C \sum_{k=1}^{\infty} 2^{k(\delta+m\beta-n/q_1-\alpha-1/2)} 2^{\epsilon_0(\delta+m\beta-n/q_1-\alpha)} \|\vec{b}\|_{\dot{\lambda}_\beta} 2^{(\epsilon_0+k)\alpha} \|f\chi_{\epsilon_0+k}\|_{L^{q_1}} \\
&\leq C 2^{\epsilon_0(-n/q_2-\alpha)} \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{\dot{K}_{q_1}^{\alpha,\infty}}.
\end{aligned}$$

For $W_2(x)$, by Hölder's inequality with $1/q'_1 + 1/q_1 = 1$, we have,

$$\begin{aligned}
W_2(x) &\leq C \left| \prod_{j=1}^m (b_j(x) - (b_j)_B) \right| \sum_{k=1}^{\infty} \int_{B_{\epsilon_0+k}} |x-z|^{-n+\delta} - |x_0-z|^{-n+\delta} |f(z)| dz \\
&\leq C \left| \prod_{j=1}^m (b_j(x) - (b_j)_B) \right| \sum_{k=1}^{\infty} \int_{B_{\epsilon_0+k}} |x-z|^{-n+\delta-1} |x-x_0| |f(z)| dz \\
&\leq C \left| \prod_{j=1}^m (b_j(x) - (b_j)_B) \right| \sum_{k=1}^{\infty} \frac{2^{\epsilon_0}}{2^{(\epsilon_0+k)(n+1-\delta)}} 2^{(\epsilon_0+k)(n-n/q_1)} \left(\int_{B_{\epsilon_0+k}} |f(z)|^{q_1} dz \right)^{1/q_1} \\
&\leq C \left| \prod_{j=1}^m (b_j(x) - (b_j)_B) \right| \sum_{k=1}^{\infty} 2^{k(\delta-n-1-\alpha+n-n/q_1)} 2^{\epsilon_0(\delta-n-\alpha+n-n/q_1)} \times \\
&\times 2^{(\epsilon_0+k)\alpha} \|f\chi_{\epsilon_0+k}\|_{L^{q_1}} \\
&\leq C 2^{\epsilon_0(\delta-\alpha-n/q_1)} \left| \prod_{j=1}^m (b_j(x) - (b_j)_B) \right| \|f\|_{\dot{K}_{q_1}^{\alpha,\infty}}.
\end{aligned}$$

Thus, by Hölder's inequality with $1/v_1 + \dots + 1/v_m = 1$, we have

$$\begin{aligned}
&2^{\epsilon_0(\alpha+n/q_2)} \left(\frac{1}{2^{\epsilon_0 n}} \int_{|x|<2^{\epsilon_0}} |W_2(x)|^{q_2} dx \right)^{1/q_2} \\
&\leq C 2^{\epsilon_0(\alpha+n/q_2)} 2^{\epsilon_0(\delta-\alpha-n/q_1)} \left(\frac{1}{2^{\epsilon_0 n}} \int_{|x|<2^{\epsilon_0}} \left| \prod_{j=1}^m (b_j(x) - (b_j)_B) \right|^{q_2} dx \right)^{1/q_2} \|f\|_{\dot{K}_{q_1}^{\alpha,\infty}} \\
&\leq C 2^{\epsilon_0(\alpha+n/q_2)} 2^{\epsilon_0(\delta-\alpha-n/q_1)} 2^{\epsilon_0(m\beta+n/q_2(1/v_1+\dots+1/v_m))} \\
&\times \left(\frac{1}{2^{\epsilon_0 n}} \frac{1}{|B|^{m\beta/n}} \prod_{j=1}^m \left(\frac{1}{|B|} \int_{|x|<2^{\epsilon_0}} |b_j(x) - (b_j)_B|^{q_2 v_j} dx \right)^{1/q_2 v_j} \right) \|f\|_{\dot{K}_{q_1}^{\alpha,\infty}} \\
&\leq C \|\vec{b}\|_{\dot{\lambda}_\beta} \|f\|_{\dot{K}_{q_1}^{\alpha,\infty}}.
\end{aligned}$$

Thus,

$$J_2 \leq C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}.$$

□

REFERENCES

- [1] S. Chanillo. A note on commutators. *Indiana Univ. Math. J.*, 31(1):7–16, 1982.
- [2] W. Chen. A Besov estimate for multilinear singular integrals. *Acta Math. Sin. (Engl. Ser.)*, 16(4):613–626, 2000.
- [3] S. Janson. Mean oscillation and commutators of singular integral operators. *Ark. Mat.*, 16(2):263–270, 1978.
- [4] L. Lanzhe. Triebel-Lizorkin space estimates for multilinear operators of sublinear operators. *Proc. Indian Acad. Sci. Math. Sci.*, 113(4):379–393, 2003.
- [5] L. Liu. Boundedness for multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces. *Extracta Math.*, 19(2):243–255, 2004.
- [6] L. Liu. Boundedness of multilinear operators on Triebel-Lizorkin spaces. *Int. J. Math. Math. Sci.*, (5-8):259–271, 2004.
- [7] L. Liu. The continuity of commutators on Triebel-Lizorkin spaces. *Integral Equations Operator Theory*, 49(1):65–75, 2004.
- [8] L.-z. Liu, S.-z. Lu, and J.-s. Xu. Boundedness for commutators of Littlewood-Paley operators. *Adv. Math. (China)*, 32(4):473–480, 2003.
- [9] S. Lu, Y. Meng, and Q. Wu. Lipschitz estimates for multilinear singular integrals. II. *Acta Math. Sci. Ser. B Engl. Ed.*, 24(2):291–300, 2004.
- [10] S. Lu, Q. Wu, and D. Yang. Boundedness of commutators on Hardy type spaces. *Sci. China Ser. A*, 45(8):984–997, 2002.
- [11] S. Z. Lu. *Four lectures on real H^p spaces*. World Scientific Publishing Co. Inc., River Edge, NJ, 1995.
- [12] S. Z. Lu and D. C. Yang. The weighted Herz-type Hardy space and its applications. *Sci. China Ser. A*, 38(6):662–673, 1995.
- [13] M. Paluszynski. Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. *Indiana Univ. Math. J.*, 44(1):1–17, 1995.
- [14] C. Pérez and R. Trujillo-González. Sharp weighted estimates for multilinear commutators. *J. London Math. Soc. (2)*, 65(3):672–692, 2002.
- [15] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
- [16] A. Torchinsky. *Real-variable methods in harmonic analysis*, volume 123 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.
- [17] D. C. Yang. The central Campanato spaces and its application. *Approx. Theory Appl. (N.S.)*, 10(4):85–99, 1994.

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COLLEGE OF MATHEMATICS,
CHANGSHA UNIVERSITY OF SCIENCE AND TECHNOLOGY,
CHANGSHA 410077, P. R. OF CHINA
E-mail address: lanzhelu@163.com