

## GENERALIZED SASAKIAN SPACE FORMS AND TRANS-SASAKIAN MANIFOLDS

G. SOMASHEKHARA AND H. G. NAGARAJA

ABSTRACT. In this paper we study the generalized Sasakian space forms and trans-Sasakian manifolds. We present results on generalized recurrent, generalized  $\phi$ -recurrent,  $\phi$ -concircular and  $\phi$ -conharmonically recurrent trans-Sasakian manifolds and generalized Sasakian space forms.

### 1. INTRODUCTION

P. Alegre, D. E. Blair and A. Carriazo [1] initiated the study of generalized Sasakian space forms and presented some examples. In [2], the authors studied the structure of generalized Sasakian space forms and proved that a K-contact generalized Sasakian space form is Sasakian and if its dimension is greater than 5 then it is a Sasakian space form. Further the authors proved that any three dimensional trans-Sasakian manifold with  $\alpha$  and  $\beta$  depending only on the direction of  $\xi$  is a generalized Sasakian space form. The authors U. C. De and Avijit Sarkar [7], [8] studied curvature properties of generalized Sasakian space forms and obtained important results. Motivated by the above studies, in this paper we study the trans-Sasakian manifolds and the generalized Sasakian space forms. The paper is organized as follows. After preliminaries in Section 2, we study generalized Sasakian space forms and generalized recurrent trans-Sasakian manifolds in Section 3. Generalized  $\phi$ -recurrent trans-Sasakian manifolds are studied in Section 4. The sections 5 and 6 contain results on  $\phi$ -concircular recurrent and  $\phi$ -conharmonically recurrent manifolds.

### 2. PRELIMINARIES

An odd dimensional Riemannian manifold  $(M, g)$  is said to be an almost contact metric manifold if there exist on  $M$  a  $(1, 1)$  tensor field  $\phi$ , a vector

---

2010 *Mathematics Subject Classification.* 53C25.

*Key words and phrases.* Generalized Sasakian space forms, trans-Sasakian,  $\phi$ -Ricci symmetric,  $\phi$ -Ricci recurrent, Concircular curvature, Einstein manifold.

field  $\xi$  and a 1-form  $\eta$  such that

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi,$$

$$(2.2) \quad (i) \eta(\xi) = 1, \quad (ii) g(X, \xi) = \eta(X),$$

$$(2.3) \quad (i) \eta(\phi X) = 0, \quad (ii) \phi\xi = 0,$$

$$(2.4) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.5) \quad g(\phi X, Y) = -g(X, \phi Y),$$

$$(2.6) \quad (\nabla_X \eta)(Y) = g(\nabla_X \xi, Y),$$

for any vector fields  $X, Y$  on  $M$ .

An almost contact metric manifold  $(M, \phi, \xi, \eta, g)$  is called a trans-Sasakian manifold [11] if there exist two functions  $\alpha$  and  $\beta$  on  $M$  such that

$$(2.7) \quad (\nabla_X \phi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for any vector fields  $X, Y$  on  $M$ . From (2.7), it follows that

$$(2.8) \quad \nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi),$$

$$(2.9) \quad (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

It is well known that  $\alpha$ -Sasakian ( $\beta = 0$ ),  $\beta$ -Kenmotsu ( $\alpha = 0$ ) and co-symplectic ( $\alpha = \beta = 0$ ) manifolds are special cases of trans-Sasakian manifolds.

A  $\phi$ -section of an almost contact metric manifold  $(M, \phi, \xi, \eta, g)$  at  $p \in M$  is a section  $\Pi \subseteq T_p M$  spanned by a unit vector  $X_p$  orthogonal to  $\xi$  and  $\phi X_p$ . The  $\phi$ -sectional curvature is defined by

$$(2.10) \quad B(X, \phi X) = R(X, \phi X, \phi X, X).$$

A Sasakian manifold with constant  $\phi$ -sectional curvature  $c$  is called a Sasakian space form and is denoted by  $M(c)$ . The Riemannian curvature tensor  $R$  in  $M(c)$  is given by

$$(2.11) \quad R(X, Y)Z = \frac{c+3}{4}R_1(X, Y)Z + \frac{c-1}{4}R_2(X, Y)Z + \frac{c-1}{4}R_3(X, Y)Z,$$

for any vector fields  $X, Y$  and  $Z$  on  $M$ , where

$$(2.12) \quad R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$

$$(2.13) \quad R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$

and

$$(2.14) \quad R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi.$$

In [1], the authors introduced the notion of generalized Sasakian space form as an almost contact metric manifold  $(M, \phi, \xi, \eta, g)$  whose Riemannian curvature tensor satisfies

$$(2.15) \quad R(X, Y)Z = f_1R_1(X, Y)Z + f_2R_2(X, Y)Z + f_3R_3(X, Y)Z,$$

where  $f_1, f_2$  and  $f_3$  are differentiable functions on  $M$ .

Throughout the paper  $M(f_1, f_2, f_3)$  denote a generalized Sasakian space form, where  $M$  is a trans-Sasakian manifold.

In a generalized Sasakian space form the following hold:

$$(2.16) \quad S(Y, Z) = [(n - 1)f_1 + 3f_2 - f_3]g(Y, Z) - [3f_2 + (n - 2)f_3]\eta(Y)\eta(Z),$$

$$(2.17) \quad QY = [(n - 1)f_1 + 3f_2 - f_3]Y - [3f_2 + (n - 2)f_3]\eta(Y)\xi,$$

$$(2.18) \quad S(Y, \xi) = (n - 1)(f_1 - f_3)\eta(Y),$$

$$(2.19) \quad Q\xi = (n - 1)(f_1 - f_3)\xi,$$

$$(2.20) \quad r = n(n - 1)f_1 + 3(n - 1)f_2 - 2(n - 1)f_3,$$

$$(2.21) \quad R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y],$$

$$(2.22) \quad R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X],$$

$$(2.23) \quad \eta(R(X, Y)Z) = (f_1 - f_3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].$$

### 3. GENERALIZED RECURRENT MANIFOLDS

A generalized Sasakian space form  $M(f_1, f_2, f_3)$  is called generalized recurrent [5], if its curvature tensor  $R$  satisfies the condition

$$(3.1) \quad (\nabla_X R)(Y, Z)(W) = A(X)R(Y, Z)W + B(X)(g(Z, W)Y - g(Y, W)Z),$$

where  $A$  and  $B$  are two 1-forms and  $B$  is non zero.

Taking  $Y = W = \xi$  in (3.1), we obtain

$$(3.2) \quad (\nabla_X R)(\xi, Z)(\xi) = A(X)R(\xi, Z)\xi + B(X)(\eta(Z)\xi - Z).$$

By definition of covariant derivative, we have

$$(3.3) \quad (\nabla_X R)(\xi, Z)\xi \\ = \nabla_X R(\xi, Z)\xi - R(\nabla_X \xi, Z)\xi - R(\xi, \nabla_X Z)\xi - R(\xi, Z)\nabla_X \xi.$$

Using (2.21), (2.22) and (2.8) in (3.3), we obtain

$$(3.4) \quad (\nabla_X R)(\xi, Z)\xi \\ = \nabla_X [(f_1 - f_3)(\eta(Z)\xi - Z)] - (f_1 - f_3)[\eta(\nabla_X Z)\xi - \nabla_X Z] \\ - (f_1 - f_3)[- \alpha \eta(Z)\phi X + \beta \eta(Z)X - \beta \eta(X)\eta(Z)\xi] \\ - (f_1 - f_3)[- \alpha g(Z, \phi X)\xi + \beta (g(Z, X)\xi - \eta(X)\eta(Z)\xi)].$$

Using (2.8) and (2.9), equation (3.4) reduces to

$$(3.5) \quad (\nabla_X R)(\xi, Z)\xi = d((f_1 - f_3))(X)(\eta(Z)\xi - Z) + (f_1 - f_3)\nabla_X Z.$$

From (3.2) and (3.5), we have

$$(3.6) \quad [d(f_1 - f_3)(X) - A(X)(f_1 - f_3) - B(X)](\eta(Z)\xi - Z) + (f_1 - f_3)\nabla_X Z = 0.$$

Taking  $Z=\xi$  in (3.6), we obtain

$$(3.7) \quad (f_1 - f_3)\nabla_X \xi = 0.$$

From (3.7), we get that  $M$  is co-symplectic provided  $f_1 \neq f_3$ .

Thus we have

**Proposition 3.1.** *A generalized recurrent  $M(f_1, f_2, f_3)$  is co-symplectic provided  $f_1 \neq f_3$ .*

As it is well known that  $M$  is generalized Ricci-recurrent [6], if its Ricci tensor  $S$ -satisfies the condition

$$(3.8) \quad (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + (n - 1)B(X)g(Y, Z),$$

where  $A$  and  $B$  are two non-zero 1-forms. By definition of covariant derivative, we have

$$(3.9) \quad (\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi).$$

Using (2.18) and (2.8) in (3.9), we have

$$(3.10) \quad (\nabla_X S)(Y, \xi) \\ = (n - 1)[d(f_1 - f_3)(X)\eta(Y) - (f_1 - f_3)(\alpha g(\phi X, Y) - \beta g(X, Y))] \\ + \alpha S(Y, \phi X) - \beta S(Y, X).$$

Taking  $Z = \xi$  in (3.8) and using (2.18), we obtain

$$(3.11) \quad (\nabla_X S)(Y, \xi) = (n - 1)[(f_1 - f_3)A(X) + B(X)]\eta(Y).$$

From (3.10) and (3.11), we have

$$(3.12) \quad (n - 1)(f_1 - f_3)[A(X)\eta(Y) + \alpha g(\phi X, Y) - \beta g(X, Y)] + (n - 1)B(X)\eta(Y) \\ - (n - 1)d(f_1 - f_3)(X)\eta(Y) - \alpha S(Y, \phi X) + \beta S(Y, X) = 0.$$

Taking  $Y = \xi$  in (3.12), we obtain

$$(3.13) \quad (f_1 - f_3)A(X) + B(X) = d(f_1 - f_3)(X).$$

If  $f_3 - f_1 = c$ , a constant then (3.13) reduces to

$$(3.14) \quad B(X) = cA(X).$$

Since  $B(X)$  is not zero, we have  $f_1 \neq f_3$ . Thus we can state that

**Proposition 3.2.** *In a generalized recurrent  $M(f_1, f_2, f_3)$ ,  $f_1 \neq f_3$  holds. Further the 1-forms  $A(X)$  and  $B(X)$  are related by (3.14).*

It is clear from (3.1) and (3.8) that, a generalized recurrent trans-Sasakian manifold is generalized Ricci-recurrent. Hence from Proposition 3.1 and Proposition 3.2 we have

**Theorem 3.3.** *A generalized recurrent  $M(f_1, f_2, f_3)$  is always a co-symplectic manifold.*

4. GENERALIZED  $\phi$ -RECURRENT MANIFOLDS

A generalized Sasakian space form  $M(f_1, f_2, f_3)$  is called generalized  $\phi$ -Ricci recurrent [3], [6] if

$$(4.1) \quad \phi^2((\nabla_X Q)(Y)) = A(X)QY + (n - 1)B(X)Y,$$

where  $Q$  is the Ricci operator,  $A(X)$  and  $B(X)$  are non-zero 1-forms. Using (2.1) and (4.1), we have

$$(4.2) \quad -\nabla_X QY - Q(\nabla_X Y) + \eta((\nabla_X Q)(Y))\xi = A(X)QY + (n - 1)B(X)Y.$$

Taking  $Y = \xi$  in (4.2) and contracting with respect to  $Z$ , we obtain

$$(4.3) \quad -g(\nabla_X Q\xi, Z) - g(Q(\nabla_X \xi), Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = A(X)g(Q\xi, Z) + (n - 1)B(X)\eta(Z).$$

Using (2.8) and (2.19) in (4.3), we have

$$(4.4) \quad (n - 1)[(f_1 - f_3)A(X) + B(X) + d(f_1 - f_3)(X)]\eta(Z) = (n - 1)(f_1 - f_3)(\alpha g(\phi X, Z) - \beta g(X, Z)) - \alpha S(\phi X, Z) + \beta S(X, Z).$$

Changing  $Z$  to  $\phi Z$  and taking  $\beta = 0$  in (4.4), we obtain

$$(4.5) \quad S(X, Z) = (n - 1)(f_1 - f_3)g(X, Z).$$

Comparing (4.5) and (2.16), we have  $(n - 2)f_3 + 3f_2 = 0$ . Changing  $Z$  to  $\phi Z$  and taking  $\alpha = 0$  in (4.4), we obtain

$$(4.6) \quad (n - 1)(f_1 - f_3)g(X, \phi Z) - S(X, \phi Z) = 0.$$

Using (2.16) in (4.6), we have  $(n - 2)f_3 + 3f_2 = 0$ .

Thus we have

**Theorem 4.1.** *In an  $\alpha$ -Sasakian (or a  $\beta$ -Kenmotsu) generalized Sasakian space form which is  $\phi$ -Ricci recurrent the relation  $(n - 2)f_3 + 3f_2 = 0$  holds.*

If  $A(X)$  and  $B(X)$  are zero in (4.1), then  $M(f_1, f_2, f_3)$  is called  $\phi$ -Ricci symmetric [6].

It is easy to see that the relation (4.6) holds in  $\phi$ -Ricci-symmetric  $\alpha$ -Sasakian (or  $\beta$ -Kenmotsu) generalized Sasakian space form.

Conversely, suppose  $(n - 2)f_3 + 3f_2 = 0$  holds in  $\phi$ -symmetric  $M(f_1, f_2, f_3)$ . Then from (2.17), we get

$$(4.7) \quad QY = (n - 1)(f_1 - f_3)Y.$$

Differentiating covariantly with respect to  $X$ , we obtain

$$(4.8) \quad (\nabla_X Q)Y = (n - 1)\nabla_X((f_1 - f_3)Y).$$

Applying  $\phi^2$  on both sides, we obtain

$$(4.9) \quad \phi^2((\nabla_X Q)Y) = (n-1)d(f_1 - f_3)(X)\phi^2 Y.$$

i.e  $M(f_1, f_2, f_3)$  is  $\phi$ -Ricci symmetric if and only if  $f_1 - f_3$  is a constant.

It follows from (2.15) that in a generalized Sasakian Space form, the  $\xi$ -sectional curvature  $K(X, \xi)$  is given by  $K(X, \xi) = R(X, \xi, X, \xi) = f_3 - f_1$ . Thus we can state that

**Theorem 4.2.** *An  $\alpha$ -Sasakian (or a  $\beta$ -Kenmotsu) generalized Sasakian space form with constant  $\xi$ -sectional curvature is  $\phi$ -Ricci symmetric if and only if  $(n-2)f_3 + 3f_2 = 0$  holds.*

#### 5. CONCIRCULAR CURVATURE TENSOR OF $M(f_1, f_2, f_3)$

The concircular curvature tensor of  $M(f_1, f_2, f_3)$  is given by

$$(5.1) \quad \tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y].$$

**Definition 5.1.**  $M(f_1, f_2, f_3)$  is said to be  $\phi$ -concircular recurrent[15] if

$$(5.2) \quad \phi^2((\nabla_W \tilde{C})(X, Y)Z) = A(W)\tilde{C}(X, Y)Z,$$

where  $A(W)$  is a nonzero 1-form.

**Definition 5.2.**  $M(f_1, f_2, f_3)$  is said to be  $\phi$ -concircular symmetric if

$$(5.3) \quad \phi^2((\nabla_W \tilde{C})(X, Y)Z) = 0.$$

Taking Covariant differentiation of (5.1), we get

$$(5.4) \quad (\nabla_W \tilde{C})(X, Y)Z = (\nabla_W R)(X, Y)Z - \frac{dr(W)}{n(n-1)}[g(Y, Z)X - g(X, Z)Y].$$

Applying  $\phi^2$  on both the sides, we get

$$(5.5) \quad \begin{aligned} \phi^2((\nabla_W \tilde{C})(X, Y)Z) \\ = \phi^2((\nabla_W R)(X, Y)Z) - \frac{dr(W)}{n(n-1)}[g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y]. \end{aligned}$$

Suppose  $M(f_1, f_2, f_3)$  is  $\phi$ -concircular recurrent. Then from (2.15) (2.1) and (5.2) in (5.5), we get

$$\begin{aligned}
 (5.6) \quad A(W)\tilde{C}(X, Y)Z &= df_1(W)[g(X, Z)Y - g(Y, Z)X] \\
 &+ df_2[g(Y, \phi Z)\phi X - 2g(X, \phi Y)\phi Z \\
 &- g(X, \phi Z)\phi Y] + f_2[g(X, \phi Z)\phi^2((\nabla_W\phi)Y) \\
 &- g(X, (\nabla_W\phi)\phi Z)\phi Y - g(Y, \phi Z)\phi^2((\nabla_W\phi)X) \\
 &+ g(Y, (\nabla_W\phi)Z)\phi X + 2g(X, \phi Y)\phi^2((\nabla_W\phi)Z) \\
 &- 2g(X, (\nabla_W\phi)Y)\phi Z] \\
 &- \frac{dr(W)}{n(n-1)}[g(Y, Z)\phi^2X - g(X, Z)\phi^2Y].
 \end{aligned}$$

Taking  $X = \xi$  in (5.6), we get

$$\begin{aligned}
 (5.7) \quad A(W)\tilde{C}(\xi, Y)Z &= -df_1(W)(g(Y, Z)\xi - \eta(Z)Y) \\
 &- f_2(\eta(\nabla_W\phi)\phi Z)\phi Y - f_2g(Y, \phi Z)\phi^2((\nabla_W\phi)\xi) \\
 &- 2f_2\eta((\nabla_W\phi)(Y))\phi Z + \frac{dr(W)}{n(n-1)}\eta(Z)\phi^2Y.
 \end{aligned}$$

Then taking  $X = \xi$  in (5.1) and using (2.15), and (2.20), we obtain

$$(5.8) \quad \tilde{C}(\xi, Y)Z = \left[ \frac{(n-2)f_3 + 3f_2}{n} \right] (g(Y, Z)\xi - \eta(Z)Y).$$

From (5.7) and (5.8), we obtain

$$\begin{aligned}
 (5.9) \quad &- A(W) \left[ \frac{(n-2)f_3 + 3f_2}{n} \right] (g(Y, Z)\xi - \eta(Z)Y) \\
 &= -df_1(W)(g(Y, Z)\xi - \eta(Z)Y) \\
 &- f_2[\alpha g(W, \phi Z) + \beta g(\phi W, \phi Z)]\phi Y \\
 &- f_2g(Y, \phi Z)[- \alpha \phi^2W - \beta \phi^3W] + \frac{dr(W)}{n(n-1)}\eta(Z)\phi^2Y \\
 &- 2f_2[\alpha(g(W, Y) - \eta(Y)\eta(W)) + \beta g(\phi W, Y)]\phi Z.
 \end{aligned}$$

Taking  $Z = \xi$  in (5.9), we obtain

$$\begin{aligned}
 (5.10) \quad &dr(W)\phi^2Y \\
 &= -n(n-1) \left[ A(W) \left[ \frac{(n-2)f_3 + 3f_2}{n} \right] - df_1(W) \right] (\eta(Y)\xi - Y).
 \end{aligned}$$

For constant  $r$ , (5.10) yields

$$(5.11) \quad ((n-2)f_3 + 3f_2)A(W) = n(df_1)(W).$$

i.e.  $df_1(W) = \left[ \frac{3f_2 - 2f_3}{n} \right] A(W)$  if and only if  $r$  is a constant.

Thus we have

**Theorem 5.3.** *A  $\phi$ -concircular recurrent  $M(f_1, f_2, f_3)$  is of constant curvature if and only if  $df_1(W) = \frac{(n-2)f_3+3f_2}{n}A(W)$ .*

If  $M(f_1, f_2, f_3)$  is  $\phi$ -concircular symmetric then  $A(W) = 0$ . From (5.10) it follows that, in a  $\phi$ -concircular symmetric  $M(f_1, f_2, f_3)$ ,  $f_1$  is constant if and only if  $r$  is a constant. Thus we can state that

**Theorem 5.4.** *A  $\phi$ -concircular symmetric  $M(f_1, f_2, f_3)$  is of constant curvature if and only if  $f_1$  is a constant.*

### 6. $\phi$ -CONHARMONICALLY RECURRENT $M(f_1, f_2, f_3)$

The conharmonic curvature tensor of  $M(f_1, f_2, f_3)$  is given by [14]

$$(6.1) \quad N(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].$$

We know that  $M(f_1, f_2, f_3)$  is said to be  $\phi$ -conharmonically recurrent if

$$(6.2) \quad \phi^2((\nabla_W N)(X, Y)Z) = A(W)N(X, Y)Z.$$

Suppose the vector fields  $X, Y$  and  $Z$  are orthogonal to  $\xi$ . Then from (2.16) (2.17) and (2.15), we have

$$(6.3) \quad S(Y, Z) = ((n-1)f_1 + 3f_2 - f_3)g(Y, Z),$$

$$(6.4) \quad QY = ((n-1)f_1 + 3f_2 - f_3)Y,$$

$$(6.5) \quad R(\xi, Y)Z = (f_1 - f_3)g(Y, Z)\xi.$$

From the equations (6.1) (6.3) and (6.4), we have

$$(6.6) \quad N(X, Y)Z = R(X, Y)Z - \frac{2}{(n-2)}[(n-1)f_1 + 3f_2 - f_3][g(Y, Z)X - g(X, Z)Y].$$

Taking the covariant derivative of (6.6) with respect to  $W$ , we get

$$(6.7) \quad (\nabla_W N)(X, Y)Z = (\nabla_W R)(X, Y)Z - \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W)(g(Y, Z)X - g(X, Z)Y).$$

Applying  $\phi^2$  on both the sides of (6.7), and using (2.15) and (6.2), we obtain

$$\begin{aligned}
 (6.8) \quad A(W)N(X, Y)Z &= df_1(W)[g(X, Z)Y - g(Y, Z)X] \\
 &+ df_2[g(Y, \phi Z)\phi X - 2g(X, \phi Y)\phi Z \\
 &- g(X, \phi Z)\phi Y] + f_2[g(X, \phi Z)\phi^2((\nabla_W\phi)Y) - g(X, (\nabla_W\phi)\phi Z)\phi Y \\
 &- g(Y, \phi Z)\phi^2((\nabla_W\phi)X) + g(Y, (\nabla_W\phi)Z)\phi X - 2g(X, (\nabla_W\phi)Y)\phi Z \\
 &+ 2g(X, \phi Y)\phi^2((\nabla_W\phi)Z)] - \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W) \\
 &\times [g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y].
 \end{aligned}$$

Taking  $X = \xi$  in (6.8), we get

$$\begin{aligned}
 (6.9) \quad A(W)N(\xi, Y)Z &= -df_1(W)g(Y, Z)\xi \\
 &- f_2\eta((\nabla_W\phi)\phi Z)\phi Y - f_2g(Y, \phi Z)\phi^2((\nabla_W\phi)\xi) - 2\eta((\nabla_W\phi)Y)\phi Z \\
 &- \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W)g(Y, Z)\xi.
 \end{aligned}$$

Taking  $X = \xi$  in (6.1), and using (6.3), (6.4) and (6.5), we obtain

$$(6.10) \quad N(\xi, Y)Z = \left[ \frac{-nf_1 - 6f_2 + 2f_3}{n-2} \right] g(Y, Z)\xi.$$

Using (2.7) and (6.10) in (6.9), we obtain

$$\begin{aligned}
 (6.11) \quad A(W) \left[ \frac{(-nf_1 - 6f_2 + 2f_3)}{(n-2)} \right] g(Y, Z)\xi &= df_1g(Y, Z)\xi \\
 &- f_2[\alpha g(W, \phi Z) + \beta g(\phi W, \phi Z)]\phi Y \\
 &- f_2g(Y, \phi Z)[- \alpha\phi^2W - \beta\phi^3W] \\
 &- 2[\alpha g(W, Y) - \eta(Y)\eta(W)] + \beta g(\phi W, Y)\phi Z \\
 &- \frac{2}{n-2}d[(n-1)f_1 + 3f_2 - f_3](W)g(Y, Z)\xi.
 \end{aligned}$$

Contracting (6.11) with respect to  $\xi$ , we get

$$d[(3n-4)f_1 + 6f_2 - 2f_3](W) = A(W)[nf_1 + 6f_2 - 2f_3].$$

Thus we can state that

**Theorem 6.1.** *In a  $\phi$ -conharmonically recurrent  $M(f_1, f_2, f_3)$ ,  $nf_1 + 6f_2 - 2f_3 = 0$  holds if and only if  $f_1$  and  $3f_2 - f_3$  are constants.*

REFERENCES

[1] P. Alegre, D. E. Blair, and A. Carriazo. Generalized Sasakian-space-forms. *Israel J. Math.*, 141:157–183, 2004.  
 [2] P. Alegre and A. Carriazo. Structures on generalized Sasakian-space-forms. *Differential Geom. Appl.*, 26(6):656–666, 2008.  
 [3] A. Başari and C. Murathan. On generalized  $\phi$ -recurrent Kenmotsu manifolds. *Fen Derg.*, 3(1):91–97, 2008.

- [4] U. C. De and N. Guha. On generalised recurrent manifolds. *Proc. Math. Soc.*, 7:7–11 (1992), 1991.
- [5] U. C. De, N. Guha, and D. Kamilya. On generalized Ricci-recurrent manifolds. *Tensor (N.S.)*, 56(3):312–317, 1995.
- [6] U. C. De and A. Sarkar. On  $\phi$ -Ricci symmetric Sasakian manifolds. *Proc. Jangjeon Math. Soc.*, 11(1):47–52, 2008.
- [7] U. C. De and A. Sarkar. On the projective curvature tensor of generalized Sasakian-space-forms. *Quaest. Math.*, 33(2):245–252, 2010.
- [8] U. C. De and A. Sarkar. Some results on generalized Sasakian-space-forms. *Thai J. Math.*, 8(1):1–10, 2010.
- [9] U. C. De, A. A. Shaikh, and S. Biswas. On  $\Phi$ -recurrent Sasakian manifolds. *Novi Sad J. Math.*, 33(2):43–48, 2003.
- [10] R. S. D. Dubey. Generalized recurrent spaces. *Indian J. Pure Appl. Math.*, 10(12):1508–1513, 1979.
- [11] J. A. Oubiña. New classes of almost contact metric structures. *Publ. Math. Debrecen*, 32(3-4):187–193, 1985.
- [12] C. Özgür. On generalized recurrent Kenmotsu manifolds. *World Applied Sciences Journal*, 2(1):9–33, 2007.
- [13] C. Özgür and M. M. Tripathi. On  $P$ -Sasakian manifolds satisfying certain conditions on the concircular curvature tensor. *Turkish J. Math.*, 31(2):171–179, 2007.
- [14] H. Singh and Q. Khan. On special weakly symmetric Riemannian manifolds. *Publ. Math. Debrecen*, 58(3):523–536, 2001.
- [15] Venkatesha and C. S. Bagewadi. On concircular  $\phi$ -recurrent LP-Sasakian manifolds. *Differ. Geom. Dyn. Syst.*, 10:312–319, 2008.

*Received November 2, 2011.*

DEPARTMENT OF MATHEMATICS,  
CENTRAL COLLEGE,  
BANGALORE UNIVERSITY,  
BANGALORE-560 001, KARNATAKA, INDIA  
*E-mail address:* hgnraj@yahoo.com