

SECOND ORDER PARALLEL TENSORS ON PARA r -SASAKIAN MANIFOLDS WITH A COEFFICIENT α

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ABSTRACT. Levy [11] had proved that a second order symmetric parallel non singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [6] has proved that second order parallel tensor in a Kaehler Space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kaehlerian metric and the fundamental 2-form. In this paper, we show that a second order symmetric parallel tensor on a para r -Sasakian manifold with a coefficient α is a constant multiple of the associated metric tensor and we have also proved that there is no non zero skew symmetric second order parallel tensor on a para r -Sasakian manifold.

1. INTRODUCTION

In 1923, Eisenhart [10] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric tensor is reducible. In 1926 Levy [11] obtained the necessary and sufficient conditions for the existence of such tensors. Sharma [13] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non-singular) tensor on an n -dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [13] that on a Sasakian manifold, there is no non zero parallel 2-form. In this paper we have defined para r -Sasakian manifolds with a coefficient α (non zero scalar function) and have proved the following two theorems:

Theorem 1.1. *On a para r -Sasakian manifold with a coefficient α , a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

Theorem 1.2. *On a para r -Sasakian manifold with a coefficient α , there is no non zero parallel 2-forms.*

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2. PRELIMINARIES

Let a C^∞ differentiable manifold M be equipped with the ring of real valued differentiable functions $\mathfrak{F}(M)$ and the module of derivations $\mathfrak{X}(M)$ and a $(1, 1)$ tensor field Φ as a linear map such that

$$\Phi: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M).$$

Let there be r (C^∞) 1-forms $A_1, A_2 \dots A_r$ and r (C^∞) contravariant vector fields $T^1, T^2 \dots T^r$ satisfying the following conditions [5]

$$(2.1) \quad A_p(T^q) = \delta_q^p \text{ where } p, q = 1, 2, \dots r$$

$$(2.2) \quad \Phi(T^p) = 0 \text{ for } p = 1, 2, \dots r$$

$$(2.3) \quad A_p(\Phi X) = 0 \text{ for } p = 1, 2, \dots r$$

for any vector field $X \in \mathfrak{X}(M)$, and

$$(2.4) \quad \Phi^2 X = X - A_p(X)T^p \text{ for } p = 1, 2, \dots r.$$

Here the summation convention is employed on repeated indices where $p = 1, 2, \dots r$. If moreover M admits a positive definite Riemannian metric g such that

$$(2.5) \quad A_p(X) = g(X, T^p), \text{ for } X \in \mathfrak{X}(M)$$

$$(2.6) \quad g(\Phi X, \Phi Y) = g(X, Y) - \sum_{p=1}^r A_p(X)A_p(Y),$$

for any vector fields X and Y . Then a manifold satisfying conditions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6) is called an *almost r -para contact structure* (Φ, A_p, T^p, g) on M .

In M the following relations hold

$$(2.7a) \quad \Phi(X, Y) = g(X, \Phi Y) = g(Y, \Phi X) = \Phi(Y, X)$$

$$(2.7b) \quad \Phi(X, T^p) = 0.$$

Definition 1. If in the almost r -para contact manifold M , the following relations

$$(2.8) \quad \Phi X = \frac{1}{\alpha} (\nabla_X T^p), \quad \Phi(X, Y) = \frac{1}{\alpha} (\nabla_X A_p(Y))$$

$$(2.9a) \quad \alpha(X) = \nabla_X \alpha$$

$$(2.9b) \quad g(X, \bar{\alpha}) = \alpha(X)$$

$$(2.10) \quad \nabla_X \Phi(Y, Z) = \alpha \left[\left\{ -g(X, Y) + \sum_{p=1}^r A_p(X)A_p(Y) \right\} A_p(Z) + \left\{ -g(X, Z) + \sum_{p=1}^r A_p(X)A_p(Z) \right\} A_p(Y) \right]$$

hold where ∇ denotes the Riemannian connection of the metric tensor g , then M is called a para r -Sasakian manifold with a coefficient α .

3. PROOFS OF THEOREM 1.1 AND 1.2

In proving Theorems 1.1 and 1.2 we need the following theorems.

Theorem 3.1. *On a para r -Sasakian manifold the following holds*

$$(3.1) \quad A_p(R(X, Y)Z) = \alpha^2[g(X, Z)A_p - g(Y, Z)A_p(X)] \\ - [\alpha(X)\Phi(Y, Z) - \alpha(Y)\Phi(X, Z)].$$

Proof. In view of (2.8), (2.9)a and (2.10) the proof follows easily. $\square$

Theorem 3.2. *For a para r -Sasakian manifold we have*

$$(3.2) \quad R(T^p, X)Y = \alpha^2[A_p(Y)X - g(X, Y)T^p] + \alpha(Y)\Phi X - \bar{\alpha}\Phi(X, Y),$$

where $g(X, \bar{\alpha}) = \alpha(X)$.

Proof. The proof follows immediately after making use of (3.1) and equation (2.9)b. $\square$

Theorem 3.3. *For a para r -Sasakian manifold the following holds*

$$(3.3) \quad R(T^p, X)T^p = \beta\Phi X + \alpha^2[X - \sum_{p=1}^r A_p(X)T^p],$$

for $p = 1, 2, \dots, r$ where $\alpha(T^p) = \beta$.

Proof. In view of equation (3.2), the proof follows in an obvious manner. $\square$

4. PROOF OF THEOREMS

Proof of Theorem (1.1). Let h denote a $(0, 2)$ tensor field on a para r -Sasakian manifold M with a coefficient α such that $\nabla h = 0$, then it follows that

$$(4.1) \quad h(R(W, X)Y, Z) + h(Y, R(W, X)Z) = 0,$$

for arbitrary vector fields X, Y, Z, W on M . We can write (4.1) as

$$g(R(W, X)Y, Z) + g(Y, R(W, X)Z) = 0.$$

Substituting $W = Y = Z = T^q$ into (4.1) we get

$$(4.2) \quad g(R(T^q, X)T^q, T^q) + g(T^q, R(T^q, X)T^q) = 0.$$

In view of theorem (3.3) the above equation becomes

$$(4.3) \quad 2\beta h(\Phi X, T^q) + 2\alpha^2 h(X, T^q) - 2\alpha^2 g(X, T^q)h(T^q, T^q) = 0$$

Simplifying (4.3) we get

$$(4.4) \quad g(X, T^q)h(T^q, T^q) - h(X, T^q) - \frac{\beta}{\alpha^2}h(\Phi X, \xi) = 0.$$

Replacing X by ΦY in (4.4) we get

$$(4.5) \quad h(\Phi Y, T^q) = \frac{\beta}{\alpha^2} [h(T^q, T^q)A_p(Y) - h(Y, T^q)].$$

Using (4.4) and (4.5) we get

$$(4.6) \quad h(T^q, T^q)A_p(Y) - h(Y, T^q) = 0$$

if $\beta^2 \neq \alpha^4$. Differentiating (4.6) covariantly with respect to Y we get

$$(4.7) \quad h(T^q, T^q)g(X, \Phi Y) + 2g(X, T^q)h(\Phi Y, T^q) - h(X, \Phi Y) = 0.$$

From the above equation and (2.8a) we obtain

$$(4.8) \quad h(T^q, T^q)g(X, \Phi Y) = h(X, \Phi Y).$$

Replacing ΦY by Y in (4.8) we get

$$(4.9) \quad h(T^q, T^q)g(X, Y) = h(X, Y).$$

In view of the fact that $h(T^q, T^q)$ is constant along any vector on M , we have proved the theorem unless $\beta^2 \neq \alpha^4$. $\square$

Proof of Theorem (1.2). Let us consider h to be a parallel 2-form on a para r -Sasakian manifold M with a coefficient α . Then putting $W = Y = T^q$ in (4.1) and using Theorem 3.3 and equations (2.1)–(2.6) we get

$$(4.10) \quad \begin{aligned} \beta h(Z, \Phi X) + \alpha^2 [h(Z, X) - h(Z, T^q)A_p(X) + h(X, T^q)A_p(Z)] \\ = h(\bar{\alpha}, T^q)\Phi(Z, X) - h(\Phi X, T^q)\alpha(Z). \end{aligned}$$

Let us define a H to be $(2, 0)$ tensor field metrically equivalent to h then contracting (4.1) with H and using (2.3)–(2.6) we obtain

$$(4.11) \quad h(\beta, T^q) = 0.$$

Substituting (4.11) in (4.10) we get

$$(4.12) \quad \begin{aligned} \beta h(Z, \Phi, X) = \alpha^2 [h(Z, X) - h(Z, T^q)A_p(X) + h(X, T^q)A_p(Z)] \\ + h(\Phi X, T^q)\alpha(Z) = 0. \end{aligned}$$

On simplifying the above equation we get

$$(4.13) \quad h(\Phi \bar{\alpha}, T^q) = 0.$$

Interchanging X and Z in (4.12) we get

$$(4.14) \quad \beta [h(Z, \Phi X) + h(X, \Phi Z)] + h(\Phi X, T^q)\alpha(Z) + h(\Phi Z, T^q)\alpha(X) = 0.$$

Replacing X by ΦY in (4.14) and making use of (2.4) and (2.6) we get

$$(4.15) \quad \begin{aligned} \beta [h(Z, Y) - h(Z, T^q)A_p(Y) + h(\Phi Y, \Phi Z)] \\ + h(Y, T^q)\alpha(Z) + h(\Phi Z, T^q)\alpha(\Phi Y) = 0. \end{aligned}$$

Using the fact that h is anti symmetric in (4.15) we obtain

$$(4.16) \quad h(Y, T^q)\alpha(Z) + h(Z, T^q)\alpha(Y) - \beta [h(Z, T^q)A_p(Y) + h(Y, T^q)A_p(Z)] \\ + h(\Phi Z, T^q)\alpha(\Phi Y) + h(\Phi Y, T^q)\alpha(\Phi Z) = 0.$$

Substituting $Y = \bar{\alpha}$ in (4.16) and making use of (4.13) and (4.11) we get

$$(4.17) \quad (\hat{\alpha} - \beta^2)h(Z, T^q) + \hat{\beta}h(\Phi Z, T^q) = 0,$$

where $\hat{\alpha} = \alpha\bar{\alpha}$ and $\hat{\beta} = \alpha(\Phi\bar{\alpha})$. Replacing Z by ΦZ in (4.17) and in view of (1.4) and (1.6) we get

$$(4.18) \quad (\beta^2 - \hat{\alpha})h(\Phi Z, T^q) = \hat{\beta}h(Z, T^q),$$

where $\beta^2 \neq \hat{\alpha}$, which in view of (4.17) becomes

$$(4.19) \quad h(Z, T^q) = 0 \text{ unless } (\hat{\beta})^2 \neq (\hat{\alpha} - \beta^2)^2.$$

Using (4.19) in (4.12) we get

$$(4.20) \quad \beta h(Z, \Phi X) + \alpha^2 h(Z, X) = 0.$$

Differentiating (4.19) covariantly along Y and using the fact that $\nabla h = 0$ we get

$$(4.21) \quad h(Z, \Phi Y) = 0.$$

In view of (4.21) and (4.20), we see that $h(Y, Z) = 0$. □

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