

ON PRINCIPAL FIBRE BUNDLE OF THE CARTESIAN PRODUCT MANIFOLD

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ABSTRACT. Differentiable principal fibre bundle have been defined and studied by Kobayashi and Nomizu [3] and many other geometers. In this paper, we study structures in the principal fibre bundle (P, M, G, π) . Hexalinear frame bundle is also studied and it has been shown that the hexalinear frame bundle is the principal fibre bundle.

1. PRELIMINARIES

Let M be a $(2n+r)$ dimensional differentiable manifold of class C^∞ . suppose there exists on M , a tensor field ϕ of type $(1, 1)$, $r(C^\infty)$ vector field $\xi_1, \xi_2, \dots, \xi_r$ and $r(C^\infty)$ 1-forms $\eta^1, \eta^2, \dots, \eta^r$ satisfying

$$(1.1) \quad \phi^2 = \lambda^2 I_{2n+r} + \eta^\alpha \otimes \xi_\alpha$$

where

$$\eta^\alpha \otimes \xi_\alpha = \Sigma \eta^\alpha \otimes \xi_\alpha.$$

Also

$$(1.2) \quad \begin{aligned} \text{(i)} \quad & \phi \xi_\alpha = 0 \\ \text{(ii)} \quad & \eta^\alpha \otimes \phi = 0 \\ \text{(iii)} \quad & \eta^\alpha(\xi_\beta) + a^2 \delta_\beta^\alpha = 0 \end{aligned}$$

where $\alpha, \beta = 1, 2, \dots, r$ and δ_β^α denotes the Kronecker delta.

Thus the manifold M in view of the equations (1.1) and (1.2) will be said to possess the general r -contact structure [7].

2010 *Mathematics Subject Classification.* 55R10.

Key words and phrases. principle fibre bundle, hexalinear frame bundle, general r -contact structure.

An example of the general r -contact structure can be given as follows. Let

$$\phi =$$

$$\begin{pmatrix} 0 & 0 & - & - & 0 & -\lambda & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & - & - & 0 & 0 & -\lambda & - & - & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & - & - & 0 & 0 & 0 & - & - & -\lambda & 0 & 0 & - & - & 0 \\ +\lambda & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\ 0 & +\lambda & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & - & - & +\lambda & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \end{pmatrix}_{(2n+r) \times (2n+r)},$$

$$\eta^1 = (0 \ 0 \ - \ - \ 0 \ 0 \ 0 \ - \ - \ 0 \ -\lambda \ 0 \ - \ - \ 0)$$

$$\eta^2 = (0 \ 0 \ - \ - \ 0 \ 0 \ 0 \ - \ - \ 0 \ 0 \ -\lambda \ - \ - \ 0)$$

$$\vdots$$

$$\eta^r = (0 \ 0 \ - \ - \ 0 \ 0 \ 0 \ - \ - \ 0 \ 0 \ 0 \ - \ - \ -\lambda)$$

$$\xi_1 = \begin{pmatrix} 0 \\ 0 \\ - \\ - \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ \lambda \\ 0 \\ - \\ - \\ 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 \\ 0 \\ - \\ - \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ 0 \\ \lambda \\ - \\ - \\ 0 \end{pmatrix}, \dots, \xi_r = \begin{pmatrix} 0 \\ 0 \\ - \\ - \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ 0 \\ 0 \\ - \\ - \\ \lambda \end{pmatrix}$$

Then it can easily shown that

$$\phi^2 = \lambda^2 I_{2n+r} + \eta^\alpha \otimes \xi_a$$

let $N(X, Y)$ be the Nijenhuis tensor of the structure. Then

$$N(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y]$$

or

$$(1.3) \quad N(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \lambda^2[X, Y] + \eta^\alpha([X, Y])\xi_\alpha$$

The structure is called normal if

$$(1.4) \quad N(X, Y) - d\eta(X, Y)\xi = 0$$

A differentiable principal fibre bundle is the set $\{P, M, G, \pi\}$ where P is a differentiable manifold, G is a Lie group such that

(i) G acts on P differentiable to the right is, there exists a differentiable map $P \times G \rightarrow P$ such that $(u, g) \rightarrow ug$, $u \in P$, $g \in G$ and $ug \in P$. Also $(ug)h = u(gh)$, $h \in G$

(ii) M is the quotient manifold P/G and the map $\pi: P \rightarrow M$ is differentiable.

(iii) For each $x \in M$ and for every coordinate neighbourhood U of x , the set $\pi^{-1}(U)$ is isomorphic to $U \times G$.

Definition. A set G is called a Lie group if G is a group as well as a differentiable manifold and two maps

(i) $G \times G \rightarrow G$ such that $(g_1, g_2) \rightarrow g_1g_2$, $g_1, g_2 \in G$ and

(ii) $G \rightarrow G$ such that $g \rightarrow g^{-1}$ are differentiable.

Example. If $\text{Gl}(n, R)$ be the set of all $n \times n$ non-singular matrices over the field of real numbers, then $\text{Gl}(n, R)$ is a group under matrix multiplication. If $g \in \text{Gl}(n, R)$ we can write $g = (g_b^a)$, $g_b^a \in R$, $a, b = 1, 2, 3, \dots, n$. These n^2 real numbers g_b^a can be treated as coordinates and induce the manifold structure in $\text{Gl}(n, R)$. The maps $\text{Gl}(n, R) \times \text{Gl}(n, R) \rightarrow \text{Gl}(n, R)$ and $\text{Gl}(n, R) \rightarrow \text{Gl}(n, R)$ are differentiable and thus $\text{Gl}(n, R)$ is a Lie group. It is called the general linear group.

2. STRUCTURES IN THE PRINCIPAL FIBRE BUNDLE

Let $\{P, M, G, \pi\}$ be the principal fibre bundle with the Lie group G and the projection map π . Let w be the connection 1-form in P . Let $\{\phi, \xi_p, \eta^p, \lambda\}$ be the general almost r -contact structure in M .

Suppose $\{\bar{\phi}, \bar{\xi}_p, \bar{\eta}^p, \lambda\}$ be the left invariant general almost r -contact structure on Lie group G . For tensor field J of type $(1, 1)$ on P , define structure on M as follows:

$$(2.1) \quad \begin{aligned} (i) \quad \pi(JX) &= \phi\pi X + \frac{1}{r} \sum \{a\bar{\eta}^p(\omega X) + b\eta^p(\pi X)\}\xi_p \\ (ii) \quad \omega(JX) &= \bar{\phi}WX + \frac{1}{r} \sum \{a^{-1}(\frac{1}{\lambda^2} - b^2)\eta^p(\pi X) - b\bar{\eta}^p(\omega X)\}\bar{\xi}_p \end{aligned}$$

where a, b are the real numbers. Then it is easy to show

$$(2.2) \quad \begin{aligned} (i) \quad \pi(J^2X) &= \pi\lambda^2X \\ (ii) \quad \omega(J^2X) &= \omega\lambda^2X \end{aligned}$$

Theorem 2.1. *For the principal fibre bundle $(\{P, M, G, \pi\})$ the $(1, 1)$ tensor field J satisfying (2.1) defines an almost GF-structure on P .*

Let $M_1, M_2, \dots, M_6$ be six C^∞ manifolds each of dimension n . If $x \in M_1, y \in M_2, z \in M_3, u \in M_4, v \in M_5, w \in M_6 \rightarrow (x, y, z, u, v, w) \in M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6$ where $M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6$ is the Cartesian product manifold of $M_1, M_2, \dots, M_6$. Let $(x^1, x^2, \dots, x^n)$ or (x^i) be local coordinate system about x in M_1 , (y^j) about y in M_2 , (z^k) about z in M_3 , $(u^l), (v^m), (w^n)$ about u, v, w in M_4, M_5 and M_6 respectively then $\{(x^i, y^j, z^k, u^l, v^m, w^n)\}$ is the local coordinate system about (x, y, z, u, v, w) is the product manifold. Let X_a be tangent vector to M_1 to x, Y_b to M_2 at y, Z_c to M_3 at z etc. Then we can write

We call the set $(x^i, y^j, z^k \dots w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n)$ the hexalinear frame at $(x^i, y^j, z^k, u^l, v^m, w^n)$ in the product manifold. Let HL be the set of all hexalinear frames at different points of the product manifold $M_1 \times M_2 \times \dots \times M_6$. It can be shown that HL is also a differentiable manifold. Let us call the set

the hexalinear frame bundle of the product manifold $M_1 \times M_2 \times \cdots \times M_6$. Now we prove the following theorem:

Proof. Let

Then

is an element of $\mathrm{HL} \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \cdots \times \mathrm{Gl}(n, R)$. We can define a map

[illegible]

such that

$$\begin{aligned} & (x^i, y^j, z^k, u^l, v^m, w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n)(P_l^a, Q_m^b, R_n^c, S_o^d, T_p^e, O_q^f) \\ & \rightarrow (x^i, y^j, z^k, u^l, v^m, w^n, X_a^i P_l^a, Y_b^j Q_m^b, Z_c^k R_n^c, U_d^l S_o^d, V_e^m T_p^e, W_f^n O_q^f). \end{aligned}$$

It can also be shown that if C is an element of product Lie group

$$(AB)C = A(BC)$$

Thus $\text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R)$ acts on HL differentiably to the right. The Cartesian product manifold $M_1 \times M_2 \times \cdots \times M_6$ is the quotient manifold

$$\text{HL} / \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R)$$

and the map $\pi: \text{HL} \rightarrow M_1 \times M_2 \times \cdots \times M_6$ is differentiable.

Suppose further that $(x^i, y^j, z^k, u^l, v^m, w^n)$ is a point of the Cartesian product manifold $M_1 \times M_2 \times \cdots \times M_6$ and let

$$U = \{(x^i, y^j, z^k, u^l, v^m, w^n) / 1 \leq i, j, k, \dots, n \leq n\}$$

be its coordinate neighbourhood. Then $\pi^{-1}(U) \subset \text{HL}$ can be expressed as

$$\pi^{-1}(U) = \{(x^i, y^j, z^k, u^l, v^m, w^n, X_a^i P_l^a, Y_b^j Q_m^b, Z_c^k R_n^c, U_d^l S_o^d, V_e^m T_p^e, W_f^n O_q^f)\}$$

We can define the map

$$\pi^{-1}(U) \rightarrow U \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R)$$

such that

$$\begin{aligned} & (x^i, y^j, z^k, u^l, v^m, w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n) \\ & \rightarrow ((x^i, y^j, z^k, u^l, v^m, w^n), (X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n)) \end{aligned}$$

which is the identity map. Since identity map is always an isomorphism so $\pi^{-1}(U)$ is isomorphic to

$$U \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R) \times \text{Gl}(n, R)$$

Thus all the conditions for hexalinear frame bundle to be the principal fibre bundle are satisfied. Hence the theorem is proved. $\square$

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Received May 20, 2012.

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