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SOME GENERAL OSTROWSKI TYPE INEQUALITIES

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Abstract. A new general Ostrowski type inequality for functions whose $(n-1)$th derivatives are continuous functions of bounded variation is established. Some special cases are discussed.

1. Introduction

In 2001, J. Pečarić and S. Varošanec in [4] have proved the following Simpson type inequality involving bounded variation:

Theorem 1. *Let $f\colon [a,b] \to \mathbf{R}$ be such that $f^{(n-1)} (n \geq 1)$ is a continuous function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}(1)\quad &\Bigg|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \\ &+ \sum_{k=5,\ k \text{ is odd}}^{n} (-1)^k \frac{2}{(k-1)!}(\frac{b-a}{2})^k(\frac{1}{k} - \frac{1}{3})f^{(k-1)}(\frac{a+b}{2})\Bigg| \\ &\leq C_n(b-a)^n \bigvee_a^b (f^{(n-1)}),\end{aligned}$$

where $C_1 = \frac{1}{3}$, $C_2 = \frac{1}{24}$, $C_3 = \frac{1}{324}$ and $C_n = \frac{1}{n!}\frac{n-3}{3\cdot 2^n}, n \geq 4$.

In 2008, the author in [2] has proved the following Ostrowski type inequality involving bounded variation:

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Theorem 2. *Let $f\colon [a,b] \to \mathbf{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$. Then for any $x \in [a,b]$ and $\theta \in [0,1]$ we have*

$$\begin{aligned}(2)\quad & \left|\int_a^b f(t)\,dt - \right.\\ & \left.(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}-(1-\theta)(x-\frac{a+b}{2})f'(x)\right]\right|\\ & \le \frac{1}{16}[4(b-x)^2-4\theta(b-a)(b-x)+\theta^2(b-a)^2\\ & \qquad\qquad + |4(b-x)^2-4\theta(b-a)(b-x)-\theta^2(b-a)^2|]\bigvee_a^b(f')\end{aligned}$$

for $a \le x \le \frac{a+b}{2}$ with $\theta \in [0,1]$, and

$$\begin{aligned}(3)\quad & \left|\int_a^b f(t)\,dt - (b-a)[(1-\theta)f(x)\right.\\ & \left. + \theta\frac{f(a)+f(b)}{2}-(1-\theta)(x-\frac{a+b}{2})f'(x)]\right|\\ & \le \frac{1}{16}[4(x-a)^2-4\theta(b-a)(x-a)+\theta^2(b-a)^2\\ & \qquad\qquad + |4(x-a)^2-4\theta(b-a)(x-a)-\theta^2(b-a)^2|]\bigvee_a^b(f')\end{aligned}$$

for $\frac{a+b}{2} < x \le b$ with $\theta \in [0,1]$.

The purpose of this paper is to derive further generalization of the above inequalities which will also lead to some interesting special cases.

2. The results

We need the following result similar to that given by C. E. M. Pearce et al. in [3]:

Lemma 1. *Let $\{P_n\}_{n\in\mathbf{N}}$ and $\{Q_n\}_{n\in\mathbf{N}}$ be two sequences of harmonic polynomials, i.e.,*

$$P_n'(t) = P_{n-1}(t),\ P_0(t) = 1, \quad t \in \mathbf{R},$$

and

$$Q_n'(t) = Q_{n-1}(t),\ Q_0(t) = 1, \quad t \in \mathbf{R}.$$

Set

$$S_n(t,x) := \begin{cases} P_n(t), & t \in [a,x],\\ Q_n(t), & t \in (x,b].\end{cases}$$

Then we have the identity

$$(4)\quad (-1)^n \int_a^b S_n(t,x)\, df^{(n-1)}(t) = \int_a^b f(t)\, dt + \sum_{k=1}^{n} (-1)^k [Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a)]$$

provided that $f\colon [a,b] \to \mathbf{R}$ *is such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$.

Proof. Observe that $S_n(t,x)$ is a piecewise continuous function of t on $[a,b]$, the Riemann-Stieltjes integral in the left-hand side of (4) is guaranteed to exist. Then (4) is not difficult to find by using the integration by parts formula for Riemann-Stieltjes integrals. □

Lemma 2. *Let* $f\colon [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$. *Then for all* $x \in [a,b]$ *and any* $\theta \in [0,1]$ *we have the identity*

$$(5)\quad (-1)^n \int_a^b K_n(t,x,\theta)\, df^{(n-1)}(t) = \int_a^b f(t)\, dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta) f(x) + \theta f(b)] - \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x),$$

where

$$(6)\qquad K_n(t,x,\theta) := \begin{cases} \frac{(t-a)^n}{n!} - \frac{\theta(b-a)(t-a)^{n-1}}{2(n-1)!}, & t \in [a,x], \\ \frac{(t-b)^n}{n!} + \frac{\theta(b-a)(t-b)^{n-1}}{2(n-1)!}, & t \in (x,b]. \end{cases}$$

Proof. The proof is immediate by using the identity (4) in Lemma 1. □

Theorem 3. *Let* $f\colon [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$. *Then for any* $x \in [a,b]$ *and* $\theta \in [0,1]$ *we have*

$$(7)\quad \Bigg| \int_a^b f(t)\, dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta) f(x) + \theta f(b)] - \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x) \Bigg| \le I_n(\theta,x) \bigvee_a^b (f^{(n-1)}),$$

where

$$I_n(\theta, x) = \begin{cases} \max\{|\frac{(x-b)^n}{n!} + \frac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}|, \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}\}, & a \le x \le \frac{a+b}{2}, \\ \max\{|\frac{(x-a)^n}{n!} - \frac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}|, \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}\}, & \frac{a+b}{2} < x \le b, \end{cases} \tag{8}$$

for $0 \le (n-1)\theta \le 1$, *and*

$$I_n(\theta, x) = \begin{cases} \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}, \\ \quad \textit{if } a \le x \le b - \frac{(n-1)\theta}{2}(b-a), \\ \max\{|\frac{(x-a)^n}{n!} - \frac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}|, |\frac{(x-b)^n}{n!} + \frac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}|\}, \\ \quad \textit{if } b - \frac{(n-1)\theta}{2}(b-a) < x < a + \frac{(n-1)\theta}{2}(b-a), \\ \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}, \\ \quad \textit{if } a + \frac{(n-1)\theta}{2}(b-a) \le x \le b \end{cases} \tag{9}$$

for $1 < (n-1)\theta \le 2$, *and*

$$I_n(\theta, x) = \max\left\{\left|\frac{(x-a)^n}{n!} - \frac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}\right|, \left|\frac{(x-b)^n}{n!} + \frac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}\right|\right\}, \tag{10}$$

for $a \le x \le b$ *with* $(n-1)\theta > 2$.

Proof. Using the identity (5) in Lemma 2, we can easily derive that

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta)f(x) + \theta f(b)]\right. \\ &- \sum_{k=1}^{n-1}\left\{\frac{(-1)^k(x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!}\right. \\ &\left.\left. - \frac{\theta(b-a)[(-1)^k(x-a)^k + (b-x)^k]}{2k!}\right\} f^{(k)}(x)\right| \le I_n(\theta, x)\bigvee_a^b(f^{(n-1)}). \end{aligned} \tag{11}$$

where

$$I_n(\theta, x) = \max_{t\in[a,b]}|K_n(t, x, \theta)|. \tag{12}$$

For brevity, we put

$$P_n(t) := (t-a)^{n-1}[t - a - \frac{n\theta}{2}(b-a)], \quad t \in [a, b],$$
$$Q_n(t) := (t-b)^{n-1}[t - b + \frac{n\theta}{2}(b-a)], \quad t \in [a, b],$$

where $\theta \in [0, 1]$. It is clear that both $P_n(t)$ and $Q_n(t)$ have only one zero in (a, b) for $0 < n\theta < 2$. Denote $t_1 = a + \frac{n\theta}{2}(b-a)$ and $t_2 = b - \frac{n\theta}{2}(b-a)$. It is easy to find that $a \le t_1 < \frac{a+b}{2} < t_2 \le b$ if and only if $0 \le n\theta < 1$ as well as $a \le t_2 \le \frac{a+b}{2} \le t_1 \le b$ if and only if $1 \le n\theta \le 2$.

By differentiation, we get

$$P_n'(t) := (t-a)^{n-2}[t-a-\frac{(n-1)\theta}{2}(b-a)], \quad t \in [a,b],$$

$$Q_n'(t) := (t-b)^{n-2}[t-b+\frac{(n-1)\theta}{2}(b-a)], \quad t \in [a,b],$$

where $\theta \in [0,1]$. It is clear that both $P_n'(t)$ and $Q_n'(t)$ have only one zero in (a,b) for $0 < (n-1)\theta < 2$. Denote $t_1^* = a+\frac{(n-1)\theta}{2}(b-a)$ and $t_2^* = b-\frac{(n-1)\theta}{2}(b-a)$. It is easy to find that $a \le t_1^* < \frac{a+b}{2} < t_2^* \le b$ if and only if $0 \le (n-1)\theta < 1$ as well as $a \le t_2^* \le \frac{a+b}{2} \le t_1^* \le b$ if and only if $1 \le (n-1)\theta \le 2$.

By differential calculus, it is easy to find that t_1^* is a minimum point of $P_n(t)$ with

$$P_n(t_1^*) = -\frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}, \tag{13}$$

as well as t_2^* is a minimum point of $Q_n(t)$ for an even n and a maximum point of $Q_n(t)$ for an odd $n > 1$ with

$$Q_n(t_2^*) = (-1)^{n-1}\frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}. \tag{14}$$

From (6), (13) and (14) it is not difficult to find that

$$\begin{aligned} &\max_{t\in[a,b]} |K_n(t,x,\theta)| \\ &= \begin{cases} \max\{|\frac{(x-b)^n}{n!} + \frac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}|, \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}\}, & a \le x \le \frac{a+b}{2}, \\ \max\{|\frac{(x-a)^n}{n!} - \frac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}|, \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}\}, & \frac{a+b}{2} < x \le b, \end{cases} \end{aligned} \tag{15}$$

for $0 \le (n-1)\theta \le 1$, and

$$\begin{aligned} &\max_{t\in[a,b]} |K_n(t,x,\theta)| \\ &= \begin{cases} \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}, \\ \qquad \text{if } a \le x \le b - \frac{(n-1)\theta}{2}(b-a), \\ \max\{|\frac{(x-a)^n}{n!} - \frac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}|, |\frac{(x-b)^n}{n!} + \frac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}|\}, \\ \qquad \text{if } b - \frac{(n-1)\theta}{2}(b-a) < x < a + \frac{(n-1)\theta}{2}(b-a), \\ \frac{(n-1)^{n-1}\theta^n(b-a)^n}{n!2^n}, \\ \qquad \text{if } a + \frac{(n-1)\theta}{2}(b-a) \le x \le b \end{cases} \end{aligned} \tag{16}$$

for $1 < (n-1)\theta \le 2$, and

$$\max_{t\in[a,b]} |K_n(t,x,\theta)| \tag{17}$$

$$= \max\left\{\left|\frac{(x-a)^n}{n!} - \frac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}\right|, \left|\frac{(x-b)^n}{n!} + \frac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}\right|\right\},$$

for $a \le x \le b$ with $(n-1)\theta > 2$.

Consequently, the inequality (7) with (8), (9) and (10) follows from (11), (12) and (15)-(17). $\square$

Remark 1. It is clear that Theorem 2 is just the special case $n = 2$ of Theorem 3.

Corollary 1. *Let the assumptions of Theorem 3 hold. Then for all $n \ge 1$ and $x \in [a, b]$ we have midpoint type inequalities*

$$\left|\int_a^b f(t)\,dt - (b-a)f(x) - \sum_{k=1}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \le \bigvee_a^b (f^{(n-1)}) \times \begin{cases} \frac{(b-x)^n}{n!}, & a \le x \le \frac{a+b}{2}, \\ \frac{(x-a)^n}{n!}, & \frac{a+b}{2} \le x \le b. \end{cases} \tag{18}$$

Proof. Letting $\theta = 0$ in (7) with (8) readily produces the result (18). $\square$

Remark 2. For $n = 1$, it is clear that (18) can be written as

$$\left|\int_a^b f(t)\,dt - (b-a)f(x)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f)$$

which was first appeared in [1].

Corollary 2. *Let the assumptions of Theorem 3 hold. Then for $n = 1, 2$ we have trapezoid inequalities*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)]\right| \le \frac{b-a}{2} \bigvee_a^b (f) \tag{19}$$

and

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)]\right| \le \frac{(b-a)^2}{8} \bigvee_a^b (f'), \tag{20}$$

and for $n \ge 3$ we have trapezoid type inequalities

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)] - \sum_{k=1}^{n-1}\left\{\frac{(-1)^k(x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!}\right.\right. \\ &\left.\left. - \frac{(b-a)[(-1)^k(x-a)^k + (b-x)^k]}{2k!}\right\} f^{(k)}(x)\right| \le \bigvee_a^b (f^{(n-1)}) \times \\ &\times \max\left\{\left|\frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!}\right|, \left|\frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!}\right|\right\}. \end{aligned} \tag{21}$$

Proof. Letting $\theta = 1$ in (7) with (8), (9) and (10) readily produces the results (19)-(21). □

Corollary 3. *Let the assumptions of Theorem 3 hold. Then for $n = 1, 2, 3, 4, 5, 6$, we have Simpson type inequalities*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)]\right| \leq \bigvee_a^b(f) \times \begin{cases} \frac{a+5b}{6} - x, & a \leq x \leq \frac{a+b}{2}, \\ x - \frac{5a+b}{6}, & \frac{a+b}{2} \leq x \leq b, \end{cases} \tag{22}$$

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)] + \frac{2(b-a)}{3}(x - \frac{a+b}{2})f'(x)\right| \leq \bigvee_a^b(f') \times \begin{cases} \frac{1}{2}(\frac{a+b}{2} - x)^2 + \frac{b-a}{3}(\frac{a+b}{2} - x) + \frac{(b-a)^2}{24}, & a \leq x \leq \frac{a+b}{2}, \\ \frac{1}{2}(x - \frac{a+b}{2})^2 + \frac{b-a}{3}(x - \frac{a+b}{2}) + \frac{(b-a)^2}{24}, & \frac{a+b}{2} \leq x \leq b, \end{cases} \tag{23}$$

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)]\right. \\ &\left. + \frac{2(b-a)}{3}\left(x - \frac{a+b}{2}\right)f'(x) - \frac{b-a}{3}\left(x - \frac{a+b}{2}\right)^2 f''(x)\right| \\ &\leq \bigvee_a^b(f'') \times \begin{cases} \frac{1}{12}(\frac{a+b}{2} - x)^3 + \frac{b-a}{12}(\frac{a+b}{2} - x)^2 & \\ \frac{(b-a)^2}{48}(\frac{a+b}{2} - x) + \frac{(b-a)^3}{648} & \\ +|\frac{1}{12}(\frac{a+b}{2} - x)^3 + \frac{b-a}{12}(\frac{a+b}{2} - x)^2 & \\ +\frac{(b-a)^2}{48}(\frac{a+b}{2} - x) - \frac{(b-a)^3}{648}|, & \text{if } a \leq x \leq \frac{a+b}{2}, \\ \frac{1}{12}(x - \frac{a+b}{2})^3 + \frac{b-a}{12}(x - \frac{a+b}{2})^2 & \\ +\frac{(b-a)^2}{48}(x - \frac{a+b}{2}) + \frac{(b-a)^3}{648} + & \\ |\frac{1}{12}(x - \frac{a+b}{2})^3 + \frac{b-a}{12}(x - \frac{a+b}{2})^2 & \\ +\frac{(b-a)^2}{48}(x - \frac{a+b}{2}) - \frac{(b-a)^3}{648}|, & \text{if } \frac{a+b}{2} \leq x \leq b, \end{cases} \end{aligned} \tag{24}$$

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)] + \frac{2(b-a)}{3}\left(x - \frac{a+b}{2}\right)f'(x)\right. \\ &\left. - \frac{b-a}{3}\left(x - \frac{a+b}{2}\right)^2 f''(x) + \frac{b-a}{9}\left(x - \frac{a+b}{2}\right)^3 f'''(x)\right| \\ &\leq \frac{(b-a)^4}{1152}\bigvee_a^b(f'''), \quad a \leq x \leq b, \end{aligned} \tag{25}$$

$$
\begin{aligned}
(26)\quad & \Big|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)] + \frac{2(b-a)}{3}(x-\frac{a+b}{2})f'(x) \\
& - \frac{b-a}{3}(x-\frac{a+b}{2})^2 f''(x) + \frac{b-a}{9}(x-\frac{a+b}{2})^3 f'''(x) \\
& + [\frac{(b-a)^5}{2880} - \frac{b-a}{36}(x-\frac{a+b}{2})^4] f^{(4)}(x)\Big| \\
& \le \bigvee_a^b (f^{(4)}) \times \begin{cases} \frac{(b-a)^5}{3645}, & \text{if } a \le x \le a+\frac{2a+b}{3}, \\ \frac{(b-a)^5}{5760} - \frac{b-a}{72}(x-\frac{a+b}{2})^4 & \\ +\frac{b-a}{8}|x-\frac{a+b}{2}||\frac{1}{15}(x-\frac{a+b}{2})^4 & \\ +\frac{b-a}{18}(x-\frac{a+b}{2})^2 - \frac{(b-a)^4}{144}|, & \text{if } \frac{2a+b}{3} < x < \frac{a+2b}{3}, \\ \frac{(b-a)^5}{3645}, & \text{if } \frac{a+2b}{3} \le x \le b, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(27)\quad & \Big|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)] \\
& + \frac{2(b-a)}{3}\left(x-\frac{a+b}{2}\right) f'(x) - \frac{b-a}{3}\left(x-\frac{a+b}{2}\right)^2 f''(x) \\
& + \frac{b-a}{9}\left(x-\frac{a+b}{2}\right)^3 f'''(x) + [\frac{(b-a)^5}{2880} - \frac{b-a}{36}\left(x-\frac{a+b}{2}\right)^4] f^{(4)}(x) \\
& - \left[\frac{(b-a)^5}{2880}\left(x-\frac{a+b}{2}\right) - \frac{b-a}{180}\left(x-\frac{a+b}{2}\right)^5\right] f^{(5)}(x)\Big| \\
\le \bigvee_a^b (f^{(5)}) \times & \begin{cases} \frac{625(b-a)^6}{6718464}, & \text{if } a \le x \le a+\frac{5a+b}{6}, \\ \frac{(b-a)^6}{46080} + \frac{(b-a)^4}{2304}(x-\frac{a+b}{2})^2 - \frac{(b-a)^2}{576}(x-\frac{a+b}{2})^4 & \\ -\frac{1}{720}(x-\frac{a+b}{2})^6 + \frac{b-a}{360}[\frac{(b-a)^4}{16} & \\ -(x-\frac{a+b}{2})^4]|x-\frac{a+b}{2}|, & \text{if } \frac{5a+b}{6} < x < \frac{a+5b}{6}, \\ \frac{625(b-a)^6}{6718464}, & \text{if } \frac{a+5b}{6} \le x \le b \end{cases}
\end{aligned}
$$

and for $n \ge 7$ with $a \le x \le b$ we have Simpson type inequalities

$$
\begin{aligned}
(28)\quad & \Big|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)] \\
& - \sum_{k=1}^{n-1}\left\{\frac{(-1)^k(x-a)^{k+1}+(b-x)^{k+1}}{(k+1)!}\right. \\
& \left. - \frac{(b-a)[(-1)^k(x-a)^k+(b-x)^k]}{6k!}\right\} f^{(k)}(x)\Big| \le \bigvee_a^b (f^{(n-1)}) \times \\
& \times \max\left\{\left|\frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{6(n-1)!}\right|, \left|\frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{6(n-1)!}\right|\right\}.
\end{aligned}
$$

Proof. Letting $\theta = \frac{1}{3}$ in (7) with (8), (9) and (10) readily produces the results (22)-(28). $\square$

Corollary 4. *Let the assumptions of Theorem 3 hold. Then for* $n = 1, 2, 3, 4,$ *we have averaged midpoint-trapezoid type inequalities*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a) + 2f(x) + f(b)]\right| \leq \bigvee_a^b (f) \times \begin{cases} \frac{a+3b}{4} - x, & \text{if } a \leq x \leq \frac{a+b}{2}, \\ x - \frac{3a+b}{4}, & \text{if } \frac{a+b}{2} \leq x \leq b, \end{cases} \tag{29}$$

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a) + 2f(x) + f(b)] + \frac{b-a}{2}\left(x - \frac{a+b}{2}\right) f'(x)\right| \leq \bigvee_a^b (f') \times \begin{cases} \frac{1}{4}(\frac{a+b}{2} - x)^2 + \frac{b-a}{8}(\frac{a+b}{2} - x) + \frac{(b-a)^2}{64} + \\ |\frac{1}{4}(\frac{a+b}{2} - x)^2 + \frac{b-a}{8}(\frac{a+b}{2} - x) - \frac{(b-a)^2}{64}|, & a \leq x \leq \frac{a+b}{2}, \\ \frac{1}{4}(x - \frac{a+b}{2})^2 + \frac{b-a}{8}(x - \frac{a+b}{2}) + \frac{(b-a)^2}{64} + \\ |\frac{1}{4}(x - \frac{a+b}{2})^2 + \frac{b-a}{8}(x - \frac{a+b}{2}) - \frac{(b-a)^2}{64}|, & \frac{a+b}{2} \leq x \leq b, \end{cases} \tag{30}$$

$$\Bigg|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a) + 2f(x) + f(b)] + \frac{b-a}{2}\left(x - \frac{a+b}{2}\right) f'(x) + \left[\frac{(b-a)^3}{48} - \frac{b-a}{4}\left(x - \frac{a+b}{2}\right)^2\right] f''(x)\Bigg| \leq \frac{(b-a)^2}{96} \bigvee_a^b (f''), \quad a \leq x \leq b \tag{31}$$

and

$$\Bigg|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a) + 2f(x) + f(b)] + \frac{b-a}{2}\left(x - \frac{a+b}{2}\right) f'(x) + \left[\frac{(b-a)^3}{48} - \frac{b-a}{4}\left(x - \frac{a+b}{2}\right)^2\right] f''(x) - \left[\frac{(b-a)^3}{48}\left(x - \frac{a+b}{2}\right) - \frac{b-a}{12}\left(x - \frac{a+b}{2}\right)^3\right] f'''(x)\Bigg|$$
$$\leq \bigvee_a^b (f''') \times \begin{cases} \frac{27(b-a)^4}{6144}, & a \leq x \leq a + \frac{3a+b}{4}, \\ \frac{(b-a)^4}{384} + \frac{(b-a)^3}{96}|x - \frac{a+b}{2}| \\ -\frac{b-a}{24}|x - \frac{a+b}{2}|^3 - \frac{1}{24}(x - \frac{a+b}{2})^4, & \frac{3a+b}{4} < x < \frac{a+3b}{4}, \\ \frac{27(b-a)^4}{6144}, & \frac{a+3b}{4} \leq x \leq b, \end{cases} \tag{32}$$

and for $n \geq 5$ with $a \leq x \leq b$ we have averaged midpoint-trapezoid type inequalities

$$
\begin{aligned}
(33)\quad & \Bigg| \int_a^b f(t)\,dt - \frac{b-a}{4}[f(a) + 2f(x) + f(b)] \\
& - \sum_{k=1}^{n-1} \Bigg\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \\
& - \frac{(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{4k!} \Bigg\} f^{(k)}(x) \Bigg| \leq \bigvee_a^b (f^{(n-1)}) \times \\
\times \max & \left\{ \left| \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{4(n-1)!} \right|, \left| \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{4(n-1)!} \right| \right\}.
\end{aligned}
$$

Proof. Letting $\theta = \frac{1}{2}$ in (7) with (8), (9) and (10) readily produces the results (29)-(33). □

Corollary 5. *Let the assumptions of Theorem 3 hold. Then for any $\theta \in [0,1]$ we have*

$$
\begin{aligned}
(34)\quad & \Bigg| \int_a^b f(x)\,dx - \frac{b-a}{2} \left[\theta f(a) + 2(1-\theta) f\left(\frac{a+b}{2}\right) + \theta f(b) \right] \\
& - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{[1-(n+1)\theta](b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \Bigg| \\
& \qquad \leq \bigvee_a^b (f^{(n-1)}) \times \begin{cases} \frac{\max\{|1-n\theta|, (n-1)^{n-1}\theta^n\}(b-a)^n}{(n!)2^n}, & n < \frac{1}{\theta} + 1, \\ \frac{(n\theta-1)(b-a)^n}{(n!)2^n}, & n \geq \frac{1}{\theta} + 1. \end{cases}
\end{aligned}
$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$.

Proof. Letting $x = \frac{a+b}{2}$ in (7) with (8), (9) and (10) readily produces the result (34). □

Remark 3. If we take $\theta = 0$ in (34), we get the midpoint type inequality

$$
\begin{aligned}
(35)\quad & \left| \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\
& \qquad \leq \frac{(b-a)^n}{n!2^n} \bigvee_a^b (f^{(n-1)}).
\end{aligned}
$$

If we take $\theta = 1$ in (34), we get the trapezoid type inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \tag{36}$$
$$\leq \bigvee_a^b (f^{(n-1)}) \times \begin{cases} \frac{b-a}{2}, & n=1, \\ \frac{(n-2)(b-a)^n}{n!2^{n+1}}, & n \geq 2. \end{cases}$$

If we take $\theta = \frac{1}{3}$ in (34), we get the Simpson type inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right. \tag{37}$$
$$\left. + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \bigvee_a^b (f^{(n-1)}) \times \begin{cases} \frac{b-a}{3}, & n=1, \\ \frac{(b-a)^2}{24}, & n=2, \\ \frac{(b-a)^3}{324}, & n=3, \\ \frac{(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 4 \end{cases}$$

which is just equal to (1) in Theorem 1.

If we take $\theta = \frac{1}{2}$ in (34), we get the averaged midpoint-trapezoid type inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right. \tag{38}$$
$$\left. + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \bigvee_a^b (f^{(n-1)}) \times \begin{cases} \frac{b-a}{4}, & n=1, \\ \frac{(b-a)^2}{32}, & n=2, \\ \frac{(n-2)(b-a)^n}{n!2^{n+1}}, & n \geq 3. \end{cases}$$

Theorem 4. *Let $f\colon [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for any $x \in [a,b]$ and $\theta \in [0,1]$ we have*

$$\begin{aligned}\Bigg| \int_a^b f(t)\,dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta)f(x) + \theta f(b)] \\ - \sum_{k=1}^{n-1} \Bigg\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \\ - \frac{\theta(b-a)[(-1)^k(x-a)^k + (b-x)^k]}{2k!} \Bigg\} f^{(k)}(x) \Bigg| \le I_n(\theta,x)\|f^{(n)}\|_1,\end{aligned}$$

where $I_n(\theta,x)$ is as given in (8)-(10) and $\|f^{(n)}\|_1 = \int_a^b |f^{(n)}(t)|\,dt$ is the usual Lebesgue norm on $L_1[a,b]$.

Proof. It is immediate from Theorem 3, since if $f^{(n-1)}$ is absolutely continuous on $[a,b]$ then we certainly have

$$\bigvee_a^b (f^{(n-1)}) = \|f^{(n)}\|_1. \qquad \square$$

Theorem 5. *Let $f\colon [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$. Then for any $x \in [a,b]$ and $\theta \in [0,1]$ we have*

$$\begin{aligned}\Bigg| \int_a^b f(t)\,dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta)f(x) + \theta f(b)] \\ - \sum_{k=1}^{n-1} \Bigg\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \\ - \frac{\theta(b-a)[(-1)^k(x-a)^k + (b-x)^k]}{2k!} \Bigg\} f^{(k)}(x) \Bigg| \le I_n(\theta,x)L(b-a),\end{aligned} \tag{39}$$

where $I_n(\theta,x)$ is as given in (8)-(10).

Proof. It is immediate from Theorem 3, since if $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ then we certainly have

$$\bigvee_a^b (f^{(n-1)}) = L(b-a). \qquad \square$$

Theorem 6. *Let $f\colon [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is monotonic on $[a,b]$. Then for any $x \in [a,b]$ and $\theta \in [0,1]$ we have*

$$\begin{aligned}(40)\quad &\Bigg| \int_a^b f(t)\,dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta)f(x) + \theta f(b)] \\ &- \sum_{k=1}^{n-1} \Bigg\{ \frac{(-1)^k(x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \\ &- \frac{\theta(b-a)[(-1)^k(x-a)^k + (b-x)^k]}{2k!} \Bigg\} f^{(k)}(x) \Bigg| \\ &\qquad\qquad \le I_n(\theta,x)|f^{(n-1)}(b) - f^{(n-1)}(a)|,\end{aligned}$$

where $I_n(\theta,x)$ is as given in (8)-(10).

Proof. It is immediate from Theorem 3, since if $f^{(n-1)}$ is monotonic on $[a,b]$ then we certainly have

$$\bigvee_a^b (f^{(n-1)}) = |f^{(n-1)}(b) - f^{(n-1)}(a)|. \qquad \square$$

Remark 4. It should be noticed that we can also get some further results similar to Corollaries 1-5 for Theorem 4-6 and so are omitted.

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