

INTEGRABILITY OF DISTRIBUTIONS ON TWO KINDS OF MANIFOLD

XIE LI AND WAN YONG

ABSTRACT. In this paper, we give some sufficient and necessary conditions for integrability of distributions on an almost Hermitian manifold and a quasi Kaehlerian manifold, and generalize Bejancu's and WanYong's research work.

1. INTRODUCTION

Let $\overline{M}$ be a real differentiable manifold. An almost complex structure on $\overline{M}$ is a tensor field J of type $(1, 1)$ on $\overline{M}$ such that at every point $x \in \overline{M}$ we have $J^2 = -I$, where I denotes the identify transformation of $T_x \overline{M}$. A manifold $\overline{M}$ endowed with an almost complex structure is called an almost complex manifold.

A Hermitian metric on an almost complex manifold $\overline{M}$ is a Riemannian metric g satisfying

$$(1.1) \quad g(JX, JY) = g(X, Y),$$

for any $X, Y \in \Gamma(T\overline{M})$.

An almost complex manifold endowed with a Hermitian metric is an almost Hermitian manifold. More, we defined the torsion tensor of J or the Nijenhuis tensor of J by

$$(1.2) \quad [J, J](X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY],$$

for any $X, Y \in \Gamma(T\overline{M})$, where $[X, Y]$ is the Lie bracket of vector fields X and Y .

2010 *Mathematics Subject Classification.* 58A30.

Key words and phrases. almost Hermitian manifold, CR-submanifold, distribution, connection, integrability.

Supported by Foundation of Department of Science and Technology of Hunan Province (No. 2010SK3023).

Definition 1.1 ([1]). An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a quasi-Kaehlerian manifold if we have

$$(1.3) \quad (\overline{\nabla}_X J)Y + (\overline{\nabla}_{JX} J)JY = 0,$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.2 ([1]). An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a Kaehlerian manifold if we have

$$(1.4) \quad \overline{\nabla}_X J = 0,$$

for any $X \in \Gamma(T\overline{M})$.

Obviously, a Kaehlerian manifold is a quasi-Kaehlerian manifold.

Let M be an m -dimensional Riemannian submanifold of an n -dimensional Riemannian manifold $\overline{M}$. We denote by $TM^\perp$ the normal bundle to M and by g both metric on M and $\overline{M}$. Also, we denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, denote by ∇ the induced connection on M , and denote by $\nabla^\perp$ the induced normal connection on M .

Then, for any $X, Y \in \Gamma(TM)$ we have

$$(1.5) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where $h: \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM^\perp)$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.5) is called the Gauss formula and h is called the second fundamental form of M .

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$ we denote by $-A_V X$ and $\nabla_X^\perp V$ the tangent part and normal part of $\overline{\nabla}_X V$ respectively. Then we have

$$(1.6) \quad \overline{\nabla}_X V = -A_V X + \nabla_X^\perp V.$$

Thus, for any $V \in \Gamma(TM^\perp)$ we have a linear operator, satisfying

$$(1.7) \quad g(A_V X, Y) = g(X, A_V Y) = g(h(X, Y), V).$$

The equation (1.6) is called the Weingarten formula.

An m -dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assigns to each point x of $\overline{M}$ an m -dimensional linear subspace D_x of $T_x \overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x \in D_x$ for each $x \in \overline{M}$. When this happens we write $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $x \in \overline{M}$ there exist m differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighbourhood of x . From now on, all distributions are supposed to be differentiable of class C^∞ .

The distribution D is said to be involutive if for all vector fields $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$. A sub-manifold M of $\overline{M}$ is said to be an integral manifold of D if for every point $x \in M$, D_x coincides with the tangent space to M at x . If there exists no integral manifold of D which contains M , then M is called a maximal integral manifold or a leaf of D . The distribution D is said to be integrable if for every $x \in \overline{M}$ there exists an integral manifold of D containing x .

Definition 1.3 ([1]). Let $\overline{M}$ be a real n -dimensional almost Hermitian manifold with almost complex structure J and with Hermitian metric g . Let M be a real m -dimensional Riemannian manifold isometrically immersed in $\overline{M}$. Then M is called a CR-submanifold of $\overline{M}$ if there exist a differentiable distribution $D: x \rightarrow D_x \subset T_x M$, on M satisfying the following conditions:

- (1) D is holomorphic, that is, $J(D_x) = D_x$, for each $x \in M$,
- (2) the complementary orthogonal distribution $D^\perp: x \rightarrow D_x^\perp \subset T_x M$, is anti-invariant, that is, $J(D_x^\perp) \subset T_x M^\perp$, for each $x \in M$.

Now let M be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold $\overline{M}$. For each vector field X tangent to M , we put

$$(1.8) \quad JX = \phi X + \omega X,$$

where ϕX and ωX are respectively the tangent part and the normal part of JX . Also, for each vector field V normal to M , we put

$$(1.9) \quad JV = BV + CV,$$

where BV and CV are respectively the tangent part and the normal part of JV .

Denote by P and Q the project morphisms of TM to D and $D^\perp$, then we have

$$(1.10) \quad \phi X = JPX,$$

and

$$(1.11) \quad \omega X = JQX,$$

for any $X \in \Gamma(TM)$.

The covariant derivative of ϕ is defined by

$$(1.12) \quad (\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y,$$

for any $X, Y \in \Gamma(TM)$. On the other hand the covariant derivative of ω is defined by

$$(1.13) \quad (\nabla_X \omega)Y = \nabla_X^\perp \omega Y - \omega \nabla_X Y,$$

for any $X, Y \in \Gamma(TM)$.

2. MAIN RESULTS

Lemma 2.1 (Frobenius[1, 3]). *Distribution D on manifold M is integrable if and only if $[X, Y] \in \Gamma(D)$, for all vector fields $X, Y \in \Gamma(D)$.*

Lemma 2.2. *Let M be a CR-sub-manifold of a quasi-Kaehlerian manifold $\overline{M}$. Then we have*

$$(2.1) \quad (\nabla_X \omega)Y = -h(X, \phi Y) + Ch(X, Y) + h(\phi X, Y) + Ch(\phi X, \phi Y) \\ + \omega \nabla_{\phi X} \phi Y - \omega A_{\omega Y} \phi X + C \nabla_{\phi X}^\perp \omega Y,$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(TM)$.

Proof. For any $X \in \Gamma(D)$ and $Y \in \Gamma(TM)$, from (1.3) we have

$$(2.2) \quad 0 = \bar{\nabla}_X JY - J\bar{\nabla}_X Y - \bar{\nabla}_{JX} Y - J\bar{\nabla}_{JX} JY.$$

By using (2.2), (1.5), (1.6) and (1.8) we get

$$(2.3) \quad \begin{aligned} 0 = & \nabla_X \phi Y + h(X, \phi Y) - A_{\omega_Y} X + \nabla_X^\perp \omega Y - J\nabla_X Y - Jh(X, Y) \\ & - \nabla_{\phi X} Y - h(\phi X, Y) - h(\phi X, Y) - J\nabla_{\phi X} \phi Y \\ & - Jh(\phi X, \phi Y) + JA_{\omega_Y} \phi X - J\nabla_{\phi X}^\perp \omega Y. \end{aligned}$$

Taking account of (2.3), (1.8) and (1.9), we obtain

$$(2.4) \quad \begin{aligned} 0 = & h(X, \phi Y) - Ch(X, Y) - h(\phi X, Y) - Ch(\phi X, \phi Y) + \nabla_X^\perp \omega Y \\ & - \omega \nabla_X Y - \omega \nabla_{\phi X} \phi Y + \omega A_{\omega_Y} \phi X - C\nabla_{\phi X}^\perp \omega Y + \nabla_X \phi Y \\ & - A_{\omega_Y} X - \phi \nabla_X Y - Bh(X, Y) - \nabla_{\phi X} Y - \phi \nabla_{\phi X} \phi Y \\ & - Bh(\phi X, \phi Y) + \phi A_{\omega_Y} \phi X - B\nabla_{\phi X}^\perp \omega Y. \end{aligned}$$

By comparing to the tangent part and the normal part in (2.4), we get

$$(2.5) \quad \begin{aligned} 0 = & \nabla_X \phi Y - A_{\omega_Y} X - \phi \nabla_X Y - Bh(X, Y) - \nabla_{\phi X} Y - \phi \nabla_{\phi X} \phi Y \\ & - Bh(\phi X, \phi Y) + \phi A_{\omega_Y} \phi X - B\nabla_{\phi X}^\perp \omega Y \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} 0 = & h(X, \phi Y) - Ch(X, Y) - h(\phi X, Y) - Ch(\phi X, \phi Y) + \nabla_X^\perp \omega Y \\ & - \omega \nabla_X Y - \omega \nabla_{\phi X} \phi Y + \omega A_{\omega_Y} \phi X - C\nabla_{\phi X}^\perp \omega Y. \end{aligned}$$

Thus (2.1) follows from (2.6) and (1.13). $\square$

Theorem 2.1. *Let M be a CR-sub-manifold of a quasi-Kaehlerian manifold $\bar{M}$. Then the distribution D is integrable if and only if*

$$(2.7) \quad \omega[\phi Y, \phi X] + 2h(X, \phi Y) - 2h(\phi X, Y) = 0,$$

for any $X, Y \in \Gamma(D)$.

Proof. For any $X, Y \in \Gamma(D)$. From (1.5) and (1.8) we have

$$(2.8) \quad \omega[X, Y] = \omega(\nabla_X Y - \nabla_Y X) = \nabla_Y^\perp \omega X - \omega \nabla_Y X - \nabla_X^\perp \omega Y + \omega \nabla_X Y.$$

By using (2.8) and (1.13) we obtain

$$(2.9) \quad \omega[X, Y] = (\nabla_Y \omega)X - (\nabla_X \omega)Y.$$

Taking account of (2.9) and (2.1) we get

$$(2.10) \quad \omega[X, Y] = \omega[\phi Y, \phi X] + 2h(X, \phi Y) - 2h(\phi X, Y).$$

According to Frobenius's Theorem, we know that the distribution D is integrable if and only if $\omega[X, Y] = 0$, for any $X, Y \in \Gamma(D)$. Taking into account (2.10), we see that the distribution D is integrable if and only if (2.7) is satisfied. $\square$

Lemma 2.3 ([4, 5]). *Let $\overline{M}$ be a quasi-Karhlerian manifold. Then we have*

$$(2.11) \quad (\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X = \frac{1}{2}J[J, J](X, Y),$$

for any $X, Y \in \Gamma(T\overline{M})$.

Lemma 2.4 ([6, 7]). *Let M be a CR-sub-manifold of a quasi-Kaehlerian manifold $\overline{M}$. Then the distribution D is integrable if and only if*

$$(2.12) \quad h(X, JY) = h(JX, Y)$$

and

$$(2.13) \quad [J, J](X, Y) \in \Gamma(D),$$

for any $X, Y \in \Gamma(D)$.

Theorem 2.2. *Let M be a CR-sub-manifold of a quasi-Kaehlerian manifold $\overline{M}$. Then the distribution D is integrable if and only if*

$$(2.14) \quad h(X, JY) = h(JX, Y)$$

and

$$(2.15) \quad 4g((\overline{\nabla}_U J)Y, JX) = g([J, J](X, U), Y) - g([J, J](Y, U), X),$$

for any $X, Y \in \Gamma(D), U \in \Gamma(D^\perp)$.

Proof. For any $X, Y \in \Gamma(D), U \in \Gamma(D^\perp)$. From (2.11) and (1.1) we have

$$(2.16) \quad \frac{1}{2}g([J, J](X, U), Y) = g((\overline{\nabla}_X J)U - (\overline{\nabla}_U J)X, JY).$$

From (2.16), (1.1) and (1.5) we get

$$(2.17) \quad \begin{aligned} \frac{1}{2}g([J, J](X, U), Y) \\ = -g(JU, h(X, JY)) + g(U, \overline{\nabla}_X Y) - g((\overline{\nabla}_U J)X, JY). \end{aligned}$$

Exchanging X with Y in (2.17) we obtain

$$(2.18) \quad \begin{aligned} \frac{1}{2}g([J, J](Y, U), X) \\ = -g(JU, h(Y, JX)) + g(U, \overline{\nabla}_Y X) - g((\overline{\nabla}_U J)Y, JX). \end{aligned}$$

On the other hand, by a direct computation we achieve

$$(2.19) \quad g((\overline{\nabla}_U J)X, JY) = -g((\overline{\nabla}_U J)Y, JX).$$

From (2.17) and (2.19) we find

$$(2.20) \quad \begin{aligned} \frac{1}{2}g([J, J](X, U), Y) \\ = -g(JU, h(X, JY)) + g(U, \overline{\nabla}_X Y) + g((\overline{\nabla}_U J)Y, JX). \end{aligned}$$

(2.20)–(2.18) follows

$$\begin{aligned}
(2.21) \quad & \frac{1}{2}g([J, J](X, U), Y) - \frac{1}{2}g([J, J](Y, U), X) \\
& = g(JU, h(Y, JX) - h(X, JY)) + g(U, [X, Y]) + 2g((\bar{\nabla}_U J)Y, JX).
\end{aligned}$$

(2.21) can be become

$$\begin{aligned}
(2.22) \quad & g([X, Y], U) = \frac{1}{2}g([J, J](X, U), Y) - \frac{1}{2}g([J, J](Y, U), X) \\
& + g(JU, h(X, JY) - h(Y, JX)) - 2g((\bar{\nabla}_U J)Y, JX).
\end{aligned}$$

Suppose D is integrable. Then from Lemma 2.4 and (2.22) we have

$$h(X, JY) = h(Y, JX)$$

and

$$0 = \frac{1}{2}g([J, J](X, U), Y) - \frac{1}{2}g([J, J](Y, U), X) - 2g((\bar{\nabla}_U J)Y, JX)$$

for any $X, Y \in \Gamma(D), U \in \Gamma(D^\perp)$, which is equivalent to (2.15).

Conversely, suppose (2.14) and (2.15) are satisfied. From (2.22), (2.14) and (2.15) we have $[X, Y] \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. By Frobenius's Theorem, we know that the distribution D is integrable. $\square$

Lemma 2.5. *Let M be a CR-sub-manifold of an almost Hermitian manifold $\bar{M}$. Then we have*

$$(2.23) \quad (\nabla_X \phi)Y = A_{\omega Y}X + Bh(X, Y) + ((\bar{\nabla}_X J)Y)^\top,$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$. From (1.5) and (1.8) we have

$$(2.24) \quad (\bar{\nabla}_X J)Y = \bar{\nabla}_X(\phi Y + \omega Y) - J(\nabla_X Y + h(X, Y)).$$

By using (2.24), (1.5), (1.6), (1.8) and (1.9) we get

$$\begin{aligned}
(2.25) \quad & (\bar{\nabla}_X J)Y = \nabla_X \phi Y + h(X, \phi Y) - A_{\omega Y}X + \nabla_X^\perp \omega Y \\
& - \phi \nabla_X Y - \omega \nabla_X Y - Bh(X, Y) - Ch(X, Y).
\end{aligned}$$

By comparying to the tangent part and the normal part in (2.25), we obtain

$$(2.26) \quad ((\bar{\nabla}_X J)Y)^\top = \nabla_X \phi Y - A_{\omega Y}X - \phi \nabla_X Y - Bh(X, Y),$$

and

$$(2.27) \quad ((\bar{\nabla}_X J)Y)^\perp = h(X, \phi Y) + \nabla_X^\perp \omega Y - \omega \nabla_X Y - Ch(X, Y).$$

Taking account of (2.26) and (1.12), (2.23) is satisfied. $\square$

Theorem 2.3. *Let M be a CR-sub-manifold of an almost Hermitian manifold $\bar{M}$. Then the distribution $D^\perp$ is integrable if and only if*

$$(2.28) \quad A_{\omega U}V - A_{\omega V}U + ((\bar{\nabla}_V J)U)^\top - ((\bar{\nabla}_U J)V)^\top = 0,$$

for any $U, V \in \Gamma(D^\perp)$.

Proof. For any $U, V \in \Gamma(D^\perp)$. From (1.5) and (1.8) we have

$$(2.29) \quad \phi[U, V] = \phi(\nabla_U V - \nabla_V U) = -\nabla_U \phi V + \phi \nabla_U V + \nabla_V \phi U - \phi \nabla_V U.$$

By using (2.29) and (1.12) we obtain

$$(2.30) \quad \phi[U, V] = (\nabla_V \phi)U - (\nabla_U \phi)V.$$

Taking account of (2.23) and (2.30) we get

$$(2.31) \quad \phi[U, V] = A_{\omega U}V + ((\bar{\nabla}_V J)U)^\top - A_{\omega V}U - ((\bar{\nabla}_U J)V)^\top.$$

According to Frobenius's Theorem, we know that the distribution $D^\perp$ is integrable if and only if $\phi[U, V] = 0$, for any $U, V \in \Gamma(D^\perp)$. Taking into account (2.31), we see that the distribution $D^\perp$ is integrable if and only if (2.28) is satisfied. $\square$

REFERENCES

- [1] A. Bejancu. *Geometry of CR-submanifolds*, volume 23 of *Mathematics and its Applications (East European Series)*. D. Reidel Publishing Co., Dordrecht, 1986.
- [2] M. Okumura. *Submanifolds of a Kaehlerian manifold and a Sasakian manifold*. Michigan State Univ., 1971.
- [3] S. Shlomo. *Lectures On Symplectic Geometry*. International Press and Tsinghua University Press, 2012.
- [4] Y. Wan. On some theorems for CR-product on a quasi kaehlerian manifold. *Journal of Changsha Univ. of Electric Power(NS)*, 12:427–430, 1997.
- [5] Y. Wan. Integrability of distribution $D \oplus \{\xi\}$ on a nearly sasakian manifold. *Journal of Changsha Univ. of Science and Technology(NS)*, 4:72–74, 2007.
- [6] Y. Wan and Q. Gao. Integrability of distribution on a CR-submanifold of a quasi kaehlerian manifold. *Journal of Changsha Univ. of Electric Power(NS)*, 13:7–15, 1998.
- [7] Y. Wan and R. Guo. Semi-invariant products of a nearly Sasakian manifold. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 27(1):119–125, 2011.
- [8] Y. Wan and Y. Liu. On nonlinear boundary value problems for functional difference equations with p -Laplacian. *Discrete Dyn. Nat. Soc.*, pages Art. ID 396840, 12, 2010.
- [9] W. Yong and D. H. Pei. Integrability of distribution $D^\perp$ on a nearly Sasakian manifold. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 25(2):271–276, 2009.

Received March 29, 2013.

DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE,
CHANGSHA UNIVERSITY OF SCIENCE AND TECHNOLOGY,
CHANGSHA, HUNAN,
P. R. CHINA
E-mail address: pkums900@163.com

DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE,
CHANGSHA UNIVERSITY OF SCIENCE AND TECHNOLOGY,
CHANGSHA, HUNAN,
P. R. CHINA
E-mail address: wanyong870901@foxmail.com