

ON THE MAXIMAL OPERATORS OF WALSH-KACZMARZ-NÖRLUND MEANS

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ABSTRACT. The main aim of this paper is to investigate $(H_p, L_{p,\infty})$ type inequalities for maximal operators of Nörlund means with monotone coefficients of one-dimensional Walsh-Kaczmarz system. By applying this results we conclude a.e. convergence of such Walsh-Kaczmarz-Nörlund means.

1. INTRODUCTION

In 1948 Šneider [22] introduced the Walsh-Kaczmarz system and showed that the inequality $\limsup_{n \rightarrow \infty} D_n^\kappa(x) / \log n \geq C > 0$ holds a.e. In 1974 Schipp [17] and Young [31] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov [21] in 1981 showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f . Gát [3] proved that, for any integrable functions, the Fejér means with respect to the Walsh-Kaczmarz system converges almost everywhere to the function. He showed that the maximal operator $\sigma^{*,\kappa}$ of Walsh-Kaczmarz-Fejér means is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. Gát's result was generalized by Simon [20], who showed that the maximal operator $\sigma^{*,\kappa}$ is of type (H_p, L_p) for $p > 1/2$. In the endpoint case $p = 1/2$ Goginava [9] (see also [5], [25] and [26]) proved that maximal operator $\sigma^{*,\kappa}$ of Walsh-Kaczmarz-Fejér means is not of type $(H_{1/2}, L_{1/2})$ and Weisz [30] showed that the following is true:

Theorem W1. The maximal operator $\sigma^{*,\kappa}$ of Walsh-Kaczmarz-Fejér means is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$.

The almost everywhere convergence of (C, α) ($0 < \alpha < 1$) means with respect Walsh-Kaczmarz system was considered by Goginava [8]. Gát and Goginava [4] proved that the following is true:

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Theorem G2. The maximal operator $\sigma^{\alpha,*,\kappa}$ of (C, α) ($0 < \alpha < 1$) means with respect Walsh-Kaczmarz system is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha),\infty}$.

Goginava and Nagy [10] proved that $\sigma^{\alpha,*,\kappa}$ is not bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha)}$.

Logarithmic means with respect to the Walsh and Vilenkin systems was studied by several authors. We mention, for instance, the papers by Simon [19], Gát [2] and Blahota, Gát [1], (see also [23]). In [16] Goginava and Nagy proved that the maximal operator $R^{*,\kappa}$ of Riesz's means is bounded from the Hardy space H_p to the space $L_{p,\infty}$, when $p > 1/2$, but is not bounded from the Hardy space H_p to the space L_p , when $0 < p \leq 1/2$. They also showed that there exists a martingale $f \in H_p$, ($0 < p \leq 1$), such that the maximal operator $L^{*,\kappa}$ of Nörlund logarithmic means is not bounded in the space L_p .

In the two dimensional case approximation properties of Nörlund and Cesàro means was considered by Nagy (see [13], [14] and [15]). The results for summability of some Nörlund means of Walsh-Fourier series can be found in [6] and [24].

The main aim of this paper is to investigate $(H_p, L_{p,\infty})$ -type inequalities for the maximal operators of Nörlund means with monotone coefficients of the one-dimensional Kaczmarz-Fourier series.

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary results of independent interest. Also these results are presented in Section 3. The detailed proofs are given in Section 4.

2. DEFINITIONS AND NOTATIONS

Now, we give a brief introduction to the theory of dyadic analysis [18]. Let $\mathbf{N}_+$ denote the set of positive integers, $\mathbf{N} := \mathbf{N}_+ \cup \{0\}$.

Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form

$$x = (x_0, x_1, \dots, x_k, \dots), \quad x_k = 0 \vee 1, \quad (k \in \mathbf{N}).$$

The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighbourhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G, \\ I_n(x) &:= I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}, \end{aligned}$$

$(x \in G, n \in \mathbf{N})$. These sets are called dyadic intervals.

Denote by $0 = (0 : i \in \mathbf{N}) \in G$ the null element of G . Let $I_n := I_n(0)$, $\overline{I_n} := G \setminus I_n$ ($n \in \mathbf{N}$). Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$, the n -th coordinate of which is 1 and the rest are zeros ($n \in \mathbf{N}$).

For $k \in \mathbf{N}$ and $x \in G$ let us denote the k th Rademacher function, by

$$r_k(x) := (-1)^{x_k}.$$

Now, define the Walsh system $w := (w_n : n \in \mathbf{N})$ on G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbf{N}).$$

If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0, 1\}$ ($i \in \mathbf{N}$), i.e. n is expressed in the number system of base 2.

Denote $|n| := \max\{j \in \mathbf{N}; n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Kaczmarz functions are defined by

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Dirichlet kernels are defined by

$$D_0 := 0, \quad D_n^\psi := \sum_{i=0}^{n-1} \psi_i, \quad (\psi = w, \text{ or } \psi = \kappa).$$

The 2^n th Dirichlet kernels have a closed form (see e.g. [18])

$$(1) \quad D_{2^n}^w(x) = D_{2^n}(x) = D_{2^n}^\kappa(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases}$$

The norm (or quasi-norm) of the spaces $L_p(G)$ and $L_{p,\infty}(G)$ are defined by

$$\|f\|_p^p := \int_G |f|^p d\mu, \quad \|f\|_{L_{p,\infty}(G)}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda), \quad (0 < p < \infty),$$

respectively.

The σ -algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k ($k \in \mathbf{N}$). Denote by $f = (f^{(n)}, n \in \mathbf{N})$ a martingale with respect to $(F_n, n \in \mathbf{N})$ (for details see, e.g. [27, 29]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} |f^{(n)}|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|, \quad (x \in G).$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n}f : n \in \mathbf{N})$ is a martingale.

If f is a martingale, then the Walsh-Kaczmarz-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}^\psi(i) = \lim_{n \rightarrow \infty} \int_G f^{(n)} \psi_i d\mu, \quad (\psi = w, \text{ or } \psi = \kappa).$$

The Walsh-Kaczmarz-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n}f : n \in \mathbf{N})$ obtained from f .

The partial sums of the Walsh-Kaczmarz-Fourier series are defined as follows:

$$S_M^\psi f := \sum_{i=0}^{M-1} \widehat{f}^\psi(i) \psi_i, \quad (\psi = w, \text{ or } \psi = \kappa).$$

Let $\{q_k : k > 0\}$ be a sequence of non-negative numbers. The n th Nörlund means for the Fourier series of f is defined by

$$(2) \quad t_n^\psi := \frac{1}{Q_n} \sum_{k=1}^n q_{n-k} S_k^\psi f, \quad (\psi = w, \text{ or } \psi = \kappa),$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

It is evident that

$$t_n^\psi f(x) = \int_G f(x+t) F_n^\psi(t) dt,$$

where

$$F_n^\psi = \frac{1}{Q_n} \sum_{k=1}^n q_{n-k} D_k^\psi.$$

Let $q_0 > 0$ and $\lim_{n \rightarrow \infty} Q_n = \infty$. The summability method (2) generated by $\{q_k : k \geq 0\}$ is regular if and only if

$$(3) \quad \lim_{n \rightarrow \infty} \frac{q_{n-1}}{Q_n} = 0.$$

It can be found in [12] (see also [11]).

The n th Fejér means of a function f is given by

$$\sigma_n^\psi f := \frac{1}{n} \sum_{k=0}^{n-1} S_k^\psi f, \quad (\psi = w, \text{ or } \psi = \kappa).$$

Fejér kernel is defined in the usual manner

$$K_n^\psi := \frac{1}{n} \sum_{k=1}^n D_k^\psi, \quad (\psi = w, \text{ or } \psi = \kappa).$$

The (C, α) -means are defined as

$$\sigma_n^{\alpha, \psi} f = \frac{1}{A_n^\alpha} \sum_{k=1}^n A_{n-k}^{\alpha-1} S_k^\psi f, \quad (\psi = w, \text{ or } \psi = \kappa),$$

where

$$(4) \quad A_0^\alpha = 0, \quad A_n^\alpha = \frac{(\alpha+1) \dots (\alpha+n)}{n!}, \quad (\alpha \neq -1, -2, \dots)$$

It is known that

$$(5) \quad A_n^\alpha \sim n^\alpha, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \quad \sum_{k=1}^n A_{n-k}^{\alpha-1} = A_n^\alpha.$$

The kernel of (C, α) -means is defined in the following way

$$K_n^{\alpha, \psi} f = \frac{1}{A_n^\alpha} \sum_{k=1}^n A_{n-k}^{\alpha-1} D_k^\psi f, \quad (\psi = w, \text{ or } \psi = \kappa).$$

The n th Riesz's logarithmic mean R_n and Nörlund logarithmic mean L_n are defined by

$$R_n^\psi f := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k^\psi f}{k}, \quad L_n^\psi f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k^\psi f}{n-k}, \quad (\psi = w, \text{ or } \psi = \kappa).$$

respectively, where

$$l_n := \sum_{k=1}^{n-1} 1/k.$$

For the martingale f we consider the following maximal operators

$$t^{*, \psi} f := \sup_{n \in \mathbb{N}} |t_n^\psi f|, \quad \sigma^{*, \psi} f := \sup_{n \in \mathbb{N}} |\sigma_n^\psi f|, \quad \sigma^{\alpha, *, \psi} f := \sup_{n \in \mathbb{N}} |\sigma_n^{\alpha, \psi} f|,$$

$$R^{*, \psi} f := \sup_{n \in \mathbb{N}} |R_n^\psi f|, \quad L^{*, \psi} f := \sup_{n \in \mathbb{N}} |L_n^\psi f|, \quad (\psi = w, \text{ or } \psi = \kappa).$$

A bounded measurable function a is p -atom, if there exists an interval I , such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

3. RESULTS

Theorem 1. a) Let sequence $\{q_k : k \geq 0\}$ be non-increasing, satisfying condition

$$(6) \quad \frac{q_0 n}{Q_n} = O(1), \text{ as } n \rightarrow \infty,$$

or non-decreasing. Then the maximal operators $t^{*,\kappa}$ of Nörlund means are bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$.

b) Let $0 < p < 1/2$ and sequence $\{q_k : k \geq 0\}$ be non-decreasing, satisfying condition

$$(7) \quad \frac{q_0}{Q_n} \geq \frac{1}{n},$$

or non-increasing. Then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{n \in \mathbb{N}} \frac{\|t_n^\kappa f\|_{L_{p,\infty}}}{\|f\|_{H_p}} = \infty.$$

Theorem 2. a) Let $0 < \alpha < 1$, sequence $\{q_k : k \geq 0\}$ be non-increasing and

$$(8) \quad \frac{q_0 n^\alpha}{Q_n} = O(1), \quad \frac{q_n - q_{n+1}}{n^{\alpha-2}} = O(1), \text{ as } n \rightarrow \infty.$$

Then the maximal operator $t^{*,\kappa}$ of Nörlund means are bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha),\infty}$.

b) Let $0 < p < 1/(1+\alpha)$, sequence $\{q_k : k \geq 0\}$ be non-increasing and

$$(9) \quad \frac{q_0}{Q_n} \geq \frac{c}{n^\alpha}, \quad 0 < \alpha \leq 1, \text{ as } n \rightarrow \infty.$$

Then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{n \in \mathbb{N}} \frac{\|t_n^\kappa f\|_{L_{p,\infty}}}{\|f\|_{H_p}} = \infty.$$

c) Let sequence $\{q_k : k \geq 0\}$ be non-increasing and

$$(10) \quad \overline{\lim}_{n \rightarrow \infty} \frac{q_0 n^\alpha}{Q_n} = \infty.$$

Then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{n \in \mathbb{N}} \frac{\|t_n^\kappa f\|_{L_{1/(1+\alpha),\infty}}}{\|f\|_{H_{1/(1+\alpha)}}} = \infty.$$

The next remark shows that conditions in (8) are sharp in the following sense:

Remark 1. The sequence $\{q_k : k \geq 0\}$ of Cesàro means σ_n^α satisfy conditions

$$(11) \quad \frac{q_0}{Q_n} \geq \frac{c}{n^\alpha}, \quad q_n - q_{n+1} \geq \frac{c}{n^{\alpha-2}}, \quad 0 < \alpha \leq 1, \text{ as } n \rightarrow \infty,$$

but they are not uniformly bounded from the martingale Hardy spaces $H_{1/(1+\alpha)}(G)$ to the space $L_{1/(1+\alpha)}(G)$.

Theorem 1 follows the following result:

Corollary 1. *Let $\{q_k = \log^{(\beta)}(k+1)^\alpha : k \geq 0\}$, where $\alpha \geq 0$, $\beta \in \mathbb{N}_+$ and $\log^{(\beta)} x = \overbrace{\log \dots \log}^{\beta \text{ times}} x$. Then the following summability method*

$$\theta_n^\kappa f = \frac{1}{Q_n} \sum_{k=1}^n \log^{(\beta)}(n-k-1)^\alpha S_k^\kappa f$$

is bounded from the Hardy space $H_{1/2}$ to the space weak $-L_{1/2}$ and is not bounded from H_p to the space weak $-L_p$, when $0 < p < 1/2$.

Analogously to Theorem 1, if we apply Abel transformation we obtain that the following is true:

Corollary 2. *The maximal operator $R^{*,\kappa}$ of Riesz's means is bounded from the Hardy space $H_{1/2}$ to the space weak $-L_{1/2}$ and is not bounded from H_p to the space weak $-L_p$, when $0 < p < 1/2$.*

By combining the first and second parts of Theorem 2 we prove that the following is true:

Corollary 3. *Let $\{q_k = k^{\alpha-1} : k \geq 0\}$, where $0 < \alpha \leq 1$. Then the following summability method*

$$L_n^{\alpha,\kappa} f = \frac{1}{Q_n} \sum_{k=1}^n (n-k)^{\alpha-1} S_k^\kappa f$$

is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space weak $-L_{1/(1+\alpha)}$ and is not bounded from H_p to the space weak $-L_p$, when $0 < p < 1/(1+\alpha)$.

By applying the second part of Theorem 2 we obtain that the following is true

Corollary 4. *The maximal operator $L^{*,\kappa}$ of Nörlund logarithmic means is not bounded from the Hardy space H_p to the space weak $-L_p$, when $0 < p < 1$.*

By using Lemma 1 we get that

Corollary 5. *Let $f \in L_1$ and $\{q_k : k \geq 0\}$ be non-decreasing or non-increasing satisfying condition (8). Then $t_n^\kappa f \rightarrow f$, a.e.*

As the consequence of corollaries 2 and 5 we conclude that

Corollary 6. *Let $f \in L_1$. Then*

$$\sigma_n^\kappa f \rightarrow f, \text{ a.e., as } n \rightarrow \infty,$$

$$R_n^\kappa f \rightarrow f, \text{ a.e., as } n \rightarrow \infty$$

and

$$\sigma_n^{\alpha,\kappa} f \rightarrow f, \text{ a.e., as } n \rightarrow \infty, \quad (0 < \alpha < 1).$$

4. SOME AUXILIARY RESULTS

Lemma 1 (see [27]). *Suppose that an operator T is σ -linear and for some $0 < p \leq 1$*

$$\int_I |Ta|^p d\mu \leq c_p < \infty,$$

for every p -atom a , where I denote the support of the atom. If T is bounded from L_∞ to L_∞ , then

$$\|Tf\|_{L_p(G)} \leq c_p \|f\|_{H_p(G)}.$$

Moreover, if $0 < p < 1$ then T is of weak type- $(1,1)$:

$$\|Tf\|_{L_{1,\infty}(G)} \leq c \|f\|_{L_1(G)}.$$

Lemma 2. *Let $2^m < n \leq 2^{m+1}$. Then*

$$\begin{aligned} Q_n F_n^w &= Q_n D_{2^m} - w_{2^m-1} \sum_{l=1}^{2^m-1} (q_{n-2^m+l} - q_{n-2^m+l+1}) l K_l^w \\ &\quad - w_{2^m-1} (2^m - 1) q_{n-1} K_{2^m-1}^w + w_{2^m} Q_{n-2^m} F_{n-2^m}^w. \end{aligned}$$

Lemma 3. *Let $0 < \alpha < 1$ and sequence $\{q_k : k \geq 0\}$ be non-increasing and satisfying condition (8). Then*

$$|F_n^w| \leq \frac{c(\alpha)}{n^\alpha} \sum_{j=0}^{|n|} 2^{j\alpha} K_{2^j}^w.$$

5. PROOFS

Proof of Lemma 2. Let $2^m < n \leq 2^{m+1}$. It is easy to show that

$$(12) \quad \sum_{l=1}^n q_{n-l} D_l^w = \sum_{l=1}^{2^m} q_{n-l} D_l^w + \sum_{l=2^m+1}^n q_{n-l} D_l^w = I + II.$$

By combining Abel transformation and following equality (see [7])

$$D_{2^m-j} = D_{2^m} - w_{2^m-1} D_j, \quad j = 1, \dots, 2^m - 1,$$

we get that

$$\begin{aligned} (13) \quad I &= \sum_{l=0}^{2^m-1} q_{n-2^m+l} D_{2^m-l}^w = \sum_{l=1}^{2^m-1} q_{n-2^m+l} D_{2^m-l}^w + q_{n-2^m} D_{2^m}^w \\ &= D_{2^m} \sum_{l=0}^{2^m-1} q_{n-2^m+l} - w_{2^m-1} \sum_{l=1}^{2^m-1} q_{n-2^m+l} D_l^w \\ &= (Q_n - Q_{n-2^m}) D_{2^m} - w_{2^m-1} \sum_{l=1}^{2^m-2} (q_{n-2^m+l} - q_{n-2^m+l+1}) l K_l^w \\ &\quad - w_{2^m-1} q_{n-1} (2^m - 1) K_{2^m-1}^w. \end{aligned}$$

Since

$$D_{j+2^m}^w = D_{2^m} + w_{2^m} D_j^w, \quad j = 1, 2, \dots, 2^m - 1$$

for II we can write that

$$(14) \quad II = \sum_{l=1}^{n-2^m} q_{n-2^m-l} D_{l+2^m}^w = Q_{n-2^m} D_{2^m} + w_{2^m} Q_{n-2^m} F_{n-2^m}^w.$$

Combining (12-14) we complete the proof of Lemma 2. $\square$

Proof of Lemma 3. Let sequence $\{q_k : k \geq 0\}$ be non-increasing. The case $q_0 n / Q_n = \bar{O}(1)$, as $n \rightarrow \infty$, will be considered separately in Theorem 1. So, we can exclude this case.

Since $0 < \alpha < 1$, we may assume that $\{q_k : k \geq 0\}$ satisfy both conditions in (8) and in addition, satisfies the following

$$\frac{Q_n}{q_0 n} = \bar{o}(1), \quad \text{as } n \rightarrow \infty.$$

It follows that

$$(15) \quad q_n = q_0 \frac{q_n n}{q_0 n} \leq q_0 \frac{Q_n}{q_0 n} = \bar{o}(1), \quad \text{as } n \rightarrow \infty.$$

By using (15) we immediately get that

$$(16) \quad q_n = \sum_{l=n}^{\infty} (q_l - q_{l+1}) \leq \sum_{l=n}^{\infty} \frac{1}{l^{2-\alpha}} \leq \frac{c}{n^{1-\alpha}}$$

and

$$(17) \quad Q_n \leq \sum_{l=0}^{n-1} q_l \leq \sum_{l=1}^n \frac{c}{l^{1-\alpha}} \leq cn^{\alpha}.$$

If we apply (16) and (17) we get that

$$(18) \quad Q_n D_{2^m} \leq 2^{\alpha(m+1)} D_{2^m} \leq c A_{2^m}^{\alpha} D_{2^m}, \quad 2^m < n \leq 2^{m+1}$$

and

$$(19) \quad (2^m - 1) q_{n-1} |K_{2^m-1}^w| \leq cn^{\alpha-1} 2^m |K_{2^m-1}^w| \leq c A_n^{\alpha-1} 2^m |K_{2^m-1}^w|,$$

where A_n^{α} is defined by (4).

Let

$$n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}, \quad n_1 > n_2 > \dots > n_r, \quad n^{(k)} = 2^{n_{k+1}} + \dots + 2^{n_r}.$$

By combining (18) and (19) we have that

$$|Q_n F_n^w| \leq c A_{n^{(0)}}^{\alpha} D_{2^{n_1}} + c \sum_{l=1}^{2^{n_1}-1} \left| A_{n^{(1)+l}}^{\alpha-2} \right| |l K_l^w| + c A_{n^{(0)}}^{\alpha-1} 2^{n_1} |K_{2^{n_1}-1}^w| + c |Q_{n^{(1)}} F_{n^{(1)}}^w|.$$

By using this process r -time we get that

$$|Q_n F_n^w| \leq c \sum_{k=1}^r \left(A_{n^{(k-1)}}^\alpha D_{2^{n_k}} + \sum_{l=1}^{2^{n_k}-1} \left| A_{n^{(k)}+l}^{\alpha-2} \right| |l K_l^w| + A_{n^{(k-1)}}^{\alpha-1} 2^{n_k} |K_{2^{n_k}-1}^w| \right).$$

The next steps of the proof is analogously to Lemma 5 of the paper [4], where is proved the analogical estimation for (C, α) means. So, we leave out the details. $\square$

Proof of Theorem 1. By using Abel transformation we obtain that

$$(20) \quad Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^n q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n$$

and

$$(21) \quad t_n^\kappa f = \frac{1}{Q_n} \left(\sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j \sigma_j^\kappa f + q_0 n \sigma_n^\kappa f \right).$$

Let sequence $\{q_k : k \geq 0\}$ be non-increasing, satisfying condition (6). Then

$$\begin{aligned} |t_n^\kappa f| &\leq \frac{1}{Q_n} \left(\sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n \right) \sigma^{*,\kappa} f \\ &= \frac{-1}{Q_n} \left(\sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j \sigma_j^\kappa f + q_0 n \sigma_n^\kappa f \right) \sigma^{*,\kappa} f + \frac{2q_0 n}{Q_n} \sigma^{*,\kappa} f \\ &\leq c \sigma^{*,\kappa} f. \end{aligned}$$

Let sequence $\{q_k : k \geq 0\}$ be non-decreasing. Then

$$\leq \frac{1}{Q_n} \left(\sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n \right) \sigma^{*,\kappa} f \leq c \sigma^{*,\kappa} f.$$

It follows that $t^{*,\kappa} f \leq c \sigma^{*,\kappa} f$. By using Theorem W1 we conclude that the maximal operators $t^{*,\kappa}$ are bounded from the martingale Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$. It follows that (see Lemma 1) $t^{*,\kappa}$ is of weak type (1,1) and $t_n^\kappa f \rightarrow f$, a.e., for all $f \in L_1$.

Let

$$f_n = D_{2^{n+1}} - D_{2^n}.$$

It is evident that

$$\widehat{f_n^\kappa}(i) = \begin{cases} 1, & \text{if } i = 2^n, \dots, 2^{n+1} - 1, \\ 0, & \text{otherwise.} \end{cases}$$

From (1) we get that

$$(22) \quad \|f_n\|_{H_p} = \|D_{2^n}\|_p \leq 1/2^{n(1/p-1)}.$$

It is easy to show that

$$\begin{aligned} |t_{2^n+1}^\kappa f_n| &= \frac{1}{Q_{2^n+1}} |q_0 S_{2^n+1}^\kappa f_n| = \frac{q_0}{Q_{2^n+1}} |D_{2^n+1}^\kappa - D_{2^n}| \\ &= \frac{q_0}{Q_{2^n+1}} |\kappa_{2^n}| = \frac{q_0}{Q_{2^n+1}}. \end{aligned}$$

Let sequence $\{q_k : k \geq 0\}$ be non-increasing. Then we automatically get that

$$(23) \quad \frac{q_0}{Q_{2^n+1}} \geq \frac{q_0}{q_0(2^n+1)} = \frac{1}{2^n+1}.$$

Under condition (7) we also have inequality (23) in the case when sequence $\{q_k : k \geq 0\}$ be non-decreasing. Hence

$$\begin{aligned} (24) \quad \frac{\|t_{2^n+1}^\kappa f_n\|_{L_{p,\infty}}}{\|f_n\|_{H_p}} &\geq \frac{\frac{cq_0}{Q_{2^n+1}} \left(\mu \left\{ x \in G : |t_{2^n+1}^\kappa f_n| \geq \frac{cq_0}{Q_{2^n+1}} \right\} \right)^{1/p}}{\|f_n\|_{H_p}} \\ &\geq \frac{cq_0 2^{n(1/p-1)}}{Q_{2^n+1}} \geq \frac{cq_0 2^{n(1/p-1)}}{2^n+1} \geq cq_0 2^{n(1/p-2)}. \end{aligned}$$

Since, $0 < p < 1/2$ so $n \rightarrow \infty$ gives our statement. $\square$

Proof of Theorem 2. Since $t^{*,\kappa}$ is bounded from L_∞ to L_∞ , by Lemma 1, the proof of theorem 2 will be complete, if we show that

$$\int_{\bar{I}_N} |t^{*,\kappa} a|^{1/(1+\alpha)} d\mu \leq c < \infty,$$

for every $1/(1+\alpha)$ -atom a , where I denotes the support of the atom.

To show boundedness of $t^{*,\kappa}$ we use the method of Gát and Goginava [4]. They proved that the maximal operator $\sigma^{\alpha,*}$ of (C, α) ($0 < \alpha < 1$) means with respect Walsh-Kaczmarz system is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha),\infty}$. Their proof was depend on the following inequality

$$|K_n^{\alpha,w}| \leq \frac{c(\alpha)}{n^\alpha} \sum_{j=0}^{|n|} 2^{j\alpha} K_{2^j}^w.$$

Since our estimation of the kernel of IV is the same, it is easy to see that the proof will be quite analogous to that of Theorem G2, so we leave out details.

By using Theorem W we also conclude that the maximal operators t^* are of weak type-(1,1) and $t_n^\kappa f \rightarrow f$, a.e., for all $f \in L_1$.

Now, we prove the second part of Theorem 2. Let $0 < p < 1/(1+\alpha)$. By combining (9), (22) and (24) we have that

$$\frac{\|t_{2^n+1}^\kappa f_n\|_{L_{p,\infty}}}{\|f_n\|_{H_p}} \geq \frac{cq_0 2^{n(1/p-1)}}{Q_{2^n+1}} \geq \frac{cq_0 2^{\alpha n} (2^n+1)^{1/p-1-\alpha}}{Q_{2^n+1}} \geq c 2^{n(1/p-1-\alpha)}$$

$\rightarrow \infty$, when $n \rightarrow \infty$.

Let us prove the third part of Theorem 2. By combining (10), (22) and (24) we have that

$$\frac{\|t_{2^n+1}^\kappa f_n\|_{L_{1/(1+\alpha),\infty}}}{\|f_n\|_{H_{1/(1+\alpha)}}} \geq \frac{cq_0 2^{n\alpha}}{Q_{2^n+1}} \rightarrow \infty, \text{ when } n \rightarrow \infty.$$

This completes the proof of Theorem 2. $\square$

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