

A SHARP GENERAL L_2 INEQUALITY OF OSTROWSKI TYPE

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ABSTRACT. A sharp general L_2 inequality of Ostrowski type is established, which provides a generalization of some previous results and gives some other interesting results as special cases.

1. INTRODUCTION

In [1] and [2], we may find the following interesting sharp trapezoid type inequality and midpoint type inequality:

Theorem 1. *Let $f: [a, b] \rightarrow \mathbf{R}$ be such that f' is absolutely continuous on $[a, b]$ and $f'' \in L_2[a, b]$. Then we have sharp inequality*

$$(1) \quad \left| \int_a^b f(t) dt - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}} \sqrt{\sigma(f'')},$$

where $\sigma(\cdot)$ is defined by

$$(2) \quad \sigma(f) = \|f\|_2^2 - \frac{1}{b-a} \left(\int_a^b f(t) dt \right)^2$$

and $\|f\|_2 := [\int_a^b f^2(t) dt]^{\frac{1}{2}}$.

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Theorem 2. *Under the assumptions of Theorem 1, we have sharp inequality*

$$(3) \quad \left| \int_a^b f(t) dt - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}[f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}} \sqrt{\sigma(f'')}.$$

In [5], the author provided a Sharp L_2 inequality of Ostrowski type as follows:

Theorem 3. *Let the assumptions of Theorem 1 hold. Then for any $\theta \in [0, 1]$ and $x \in [a, b]$ we have sharp inequality*

$$(4) \quad \left| \int_a^b f(t)dt - (b-a) \left[(1-\theta)f(x) + \theta \frac{f(a)+f(b)}{2} \right] + (1-\theta)(b-a) \left(x - \frac{a+b}{2} \right) f'(x) - \left[\frac{1-\theta}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{1-3\theta}{24}(b-a)^2 \right] [f'(b) - f'(a)] \right| \leq \left[\frac{\theta(1-\theta)}{4}(b-a) \left(x - \frac{a+b}{2} \right)^4 + \frac{3\theta^2 - 5\theta + 2}{24}(b-a)^3 \left(x - \frac{a+b}{2} \right)^2 + \frac{15\theta^2 - 15\theta + 4}{2880}(b-a)^5 \right]^{\frac{1}{2}} \sqrt{\sigma(f'')}.$$

The inequality (4) not only provide a generalization of inequalities (1) and (3), but also give some other interesting sharp inequalities as special cases. Moreover, it has been shown that the corrected Simpson rule (see [8],[11] and [10]) gives better result than Simpson rule and, in particular, the corrected averaged midpoint-trapezoid quadrature rule is optimal.

In this work, we will derive a new sharp general inequality of Ostrowski type for functions whose $(n-1)$ th derivatives are absolutely continuous and whose n th derivatives belong to $L_2(a, b)$. This will not only provide a generalization of the inequality (4), but also give some other interesting sharp inequalities as special cases.

2. THE RESULTS

In [6], we may find the identity

$$\begin{aligned}
 (5) \quad & (-1)^n \int_a^b K_n(t, x, \theta) f^{(n)}(t) dt \\
 &= \int_a^b f(t) dt - \frac{b-a}{2} [\theta f(a) + 2(1-\theta)f(x) + \theta f(b)] \\
 &\quad - \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \right. \\
 &\quad \left. - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x)
 \end{aligned}$$

where $K_n(t, x, \theta)$ is the kernel given by

$$(6) \quad K_n(t, x, \theta) = \begin{cases} \frac{(t-a)^n}{n!} - \frac{\theta(b-a)(t-a)^{n-1}}{2(n-1)!}, & \text{if } t \in [a, x], \\ \frac{(t-b)^n}{n!} + \frac{(b-a)(t-b)^{n-1}}{2(n-1)!}, & \text{if } t \in (x, b]. \end{cases}$$

By elementary calculus, it is not difficult to get

$$\begin{aligned}
 (7) \quad \int_a^b K_n(t, x, \theta) dt &= \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} \\
 &\quad - \frac{\theta(b-a)[(x-a)^n + (-1)^n (b-x)^n]}{2n!}
 \end{aligned}$$

and

$$\begin{aligned}
 (8) \quad \int_a^b K_n^2(t, x, \theta) dt &= \frac{(x-a)^{2n+1} + (b-x)^{2n+1}}{(2n+1)(n!)^2} \\
 &\quad - \frac{\theta(b-a)[(x-a)^{2n} + (b-x)^{2n}]}{2(n!)^2} + \frac{\theta^2(b-a)^2[(x-a)^{2n-1} + (b-x)^{2n-1}]}{4(2n-1)[(n-1)!]^2}.
 \end{aligned}$$

For brevity, in what follows, we will use the notations

$$\begin{aligned}
 (9) \quad G_n(x, \theta) &:= \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \right. \\
 &\quad \left. - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x),
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad H_n(x, \theta) &:= \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} \\
 &\quad - \frac{\theta(b-a)[(x-a)^n + (-1)^n (b-x)^n]}{2n!},
 \end{aligned}$$

$$(11) \quad I_n(x, \theta) := \frac{(x-a)^{2n+1} + (b-x)^{2n+1}}{(2n+1)(n!)^2} - \frac{\theta(b-a)[(x-a)^{2n} + (b-x)^{2n}]}{2(n!)^2} + \frac{\theta^2(b-a)^2[(x-a)^{2n-1} + (b-x)^{2n-1}]}{4(2n-1)[(n-1)!]^2}$$

and

$$(12) \quad D_n := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}.$$

Theorem 4. Let $f: [a, b] \rightarrow \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and $f^{(n)} \in L_2[a, b]$. Then for any $\theta \in [0, 1]$ and $x \in [a, b]$ we have

$$(13) \quad \left| (-1)^n \left\{ \int_a^b f(t) dt - (b-a) \left[(1-\theta)f(x) + \theta \frac{f(a) + f(b)}{2} \right] - G_n(x, \theta) \right\} - D_n H_n(x, \theta) \right| \leq \left[I_n(x, \theta) - \frac{H_n^2(x, \theta)}{b-a} \right]^{\frac{1}{2}} \sqrt{\sigma(f^{(n)})}.$$

The inequality (13) is sharp.

Proof. From (5),(7)-(10),(12) we get

$$(14) \quad \int_a^b \left[K_n(t, x, \theta) - \frac{1}{b-a} \int_a^b K_n(s, x, \theta) ds \right] \left[f^{(n)}(t) - \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right] dt \\ = (-1)^n \left\{ \int_a^b f(t) dt - (b-a) \left[(1-\theta)f(x) + \theta \frac{f(a) + f(b)}{2} \right] - G_n(\theta, x) \right\} - D_n H_n(\theta, x).$$

By using the Cauchy-Schwarz-Bunyakowski inequality, we have

$$(15) \quad \left| \int_a^b \left[K_n(t, x, \theta) - \frac{1}{b-a} \int_a^b K_n(s, x, \theta) ds \right] \left[f^{(n)}(t) - \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right] dt \right| \\ \leq \left\| K(\cdot, x, \theta) - \frac{1}{b-a} \int_a^b K_n(s, x, \theta) ds \right\|_2 \left\| f^{(n)} - \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right\|_2.$$

From (7),(8),(10),(11) we also have

$$(16) \quad \left\| K(\cdot, x, \theta) - \frac{1}{b-a} \int_a^b K_n(t, s, \theta) ds \right\|_2^2 = I_n(\theta, x) - \frac{H_n^2(\theta, x)}{b-a}.$$

and by (2),

$$(17) \quad \left\| f^{(n)} - \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right\|_2^2 \\ = \|f^{(n)}\|_2^2 - \frac{(f^{(n-1)}(b) - f^{(n-1)}(a))^2}{b-a} = \sigma(f^{(n)}).$$

Consequently, the inequality (13) follows from (14)-(17).

It is clear that the inequality (13) is sharp, since in its proof only Cauchy-Schwarz-Bunyakowski inequality is used, and so the equality condition follows from the well-known inequality. $\square$

Remark 1. If we take $n = 1$ in Theorem 4, then we have

$$\begin{aligned} G_1(x, \theta) &= 0, \\ H_1(x, \theta) &= (1 - \theta)(b - a) \left(x - \frac{a + b}{2} \right), \\ I_1(x, \theta) &= \frac{1 - 3\theta + 3\theta^2}{12} (b - a)^3 + (1 - \theta)(b - a) \left(x - \frac{a + b}{2} \right)^2, \end{aligned}$$

and

$$I_1(x, \theta) - \frac{H_1^2(x, \theta)}{b - a} = \theta(1 - \theta)(b - a) \left(x - \frac{a + b}{2} \right)^2 + \frac{1 - 3\theta + 3\theta^2}{12} (b - a)^3.$$

Thus we can derived a sharp L_2 inequality of Ostrowski type as

$$\begin{aligned} & \left| (b - a) \left[(1 - \theta)f(x) + \theta \frac{f(a) + f(b)}{2} \right] - (1 - \theta) \left(x - \frac{a + b}{2} \right) [f(b) - f(a)] \right. \\ & \quad \left. - \int_a^b f(t) dt \right| \\ & \leq \left[\theta(1 - \theta)(b - a) \left(x - \frac{a + b}{2} \right)^2 + \frac{1 - 3\theta + 3\theta^2}{12} (b - a)^3 \right]^{\frac{1}{2}} \sqrt{\sigma(f')}, \end{aligned}$$

which has been first proved in Theorem 1 of [4].

Remark 2. If we take $n = 2$ in Theorem 4, then we have

$$\begin{aligned} G_2(x, \theta) &= (1 - \theta)(b - a) \left(\frac{a + b}{2} - x \right) f'(x), \\ H_2(x, \theta) &= \frac{(1 - 3\theta)(b - a)^3}{24} + \frac{(1 - \theta)(b - a)}{2} \left(x - \frac{a + b}{2} \right)^2, \\ I_2(x, \theta) &= \frac{3 - 15\theta + 20\theta^2}{960} (b - a)^5 + \frac{1 - 3\theta + 2\theta^2}{8} (b - a)^3 \left(x - \frac{a + b}{2} \right)^2 \\ & \quad - \frac{1 - \theta}{4} (b - a) \left(x - \frac{a + b}{2} \right)^4, \end{aligned}$$

and

$$I_2(x, \theta) - \frac{H_2^2(x, \theta)}{b-a} = \frac{\theta(1-\theta)}{4}(b-a) \left(x - \frac{a+b}{2}\right)^4 + \frac{3\theta^2 - 5\theta + 2}{24}(b-a)^3 \left(x - \frac{a+b}{2}\right)^2 + \frac{15\theta^2 - 15\theta + 4}{2880}(b-a)^5$$

Thus we can derived a sharp L_2 inequality of Ostrowski type as

$$\begin{aligned} & \left| \int_a^b f(t)dt - (b-a) \left[(1-\theta)f(x) + \theta \frac{f(a)+f(b)}{2} \right] \right. \\ & \quad \left. + (1-\theta)(b-a) \left(x - \frac{a+b}{2}\right) f'(x) \right. \\ & \quad \left. - \left[\frac{1-\theta}{2} \left(x - \frac{a+b}{2}\right)^2 + \frac{1-3\theta}{24}(b-a)^2 \right] [f'(b) - f'(a)] \right| \\ & \leq \left[\frac{\theta(1-\theta)}{4}(b-a) \left(x - \frac{a+b}{2}\right)^4 + \frac{3\theta^2 - 5\theta + 2}{24}(b-a)^3 \left(x - \frac{a+b}{2}\right)^2 \right. \\ & \quad \left. + \frac{15\theta^2 - 15\theta + 4}{2880}(b-a)^5 \right]^{\frac{1}{2}} \sqrt{\sigma(f'')}. \end{aligned}$$

which is just the inequality (4).

Corollary 1. *Let the assumptions of Theorem 4 hold. Then for any $\theta \in [0, 1]$, we get a sharp inequality*

$$\begin{aligned} (18) \quad & \left| (-1)^n \left\{ \int_a^b f(t)dt - (b-a) \left[(1-\theta)f(x) + \theta \frac{f(a)+f(b)}{2} \right] \right. \right. \\ & \quad \left. - \sum_{k=1}^{n-1} \frac{[(-1)^k + 1][1 - (k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}(x) \right. \\ & \quad \left. \left. - \frac{[1 + (-1)^n][1 - (n+1)\theta](b-a)^{n+1}}{2^{n+1}(n+1)!} D_n \right\} \right| \\ & \leq \frac{c(n, \theta)}{(n+1)!} \left(\frac{b-a}{2} \right)^{n+\frac{1}{2}} \sqrt{\sigma(f^{(n)})}, \end{aligned}$$

where $c(n, \theta) = \left[\frac{2(n+1)^2[(2n-1)-(4n^2-1)\theta + (2n+1)n^2\theta^2] - [1+(-1)^n](4n^2-1)[1-(n+1)\theta]^2}{4n^2-1} \right]^{\frac{1}{2}}$.

Proof. We set $x = \frac{a+b}{2}$ in (4) and observe that

$$\begin{aligned} G_n\left(\frac{a+b}{2}, \theta\right) &= \sum_{k=1}^{n-1} \left\{ \frac{[(-1)^k + 1](b-a)^{k+1}}{2^{k+1}(k+1)!} - \frac{\theta[(-1)^k + 1](b-a)^{k+1}}{2^{k+1}k!} \right\} f^{(k)}(x) \\ &= \sum_{k=1}^{n-1} \frac{[(-1)^k + 1][1 - (k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}(x), \end{aligned}$$

$$\begin{aligned} H_n \left(\frac{a+b}{2}, \theta \right) &= \frac{[1 + (-1)^n](b-a)^{n+1}}{2^{n+1}(n+1)!} - \frac{[1 + (-1)^n]\theta(b-a)^{n+1}}{2^{n+1}n!} \\ &= \frac{[1 + (-1)^n][1 - (n+1)\theta](b-a)^{n+1}}{2^{n+1}(n+1)!}, \end{aligned}$$

$$\begin{aligned} I_n \left(\frac{a+b}{2}, \theta \right) &= \frac{(b-a)^{2n+1}}{2^{2n}(2n+1)(n!)^2} - \frac{\theta(b-a)^{2n+1}}{2^{2n}(n!)^2} + \frac{\theta^2(b-a)^{2n+1}}{2^{2n}(2n-1)[(n-1)!]^2} \\ &= \frac{(2n-1) - (4n^2-1)\theta + (2n+1)n^2\theta^2}{2^{2n}(4n^2-1)(n!)^2} (b-a)^{2n+1}, \end{aligned}$$

and

$$I_n \left(\frac{a+b}{2}, \theta \right) - \frac{H_n^2 \left(\frac{a+b}{2}, \theta \right)}{b-a} = \frac{c^2(n, \theta)}{2^{2n+1}[(n+1)!]^2} (b-a)^{2n+1},$$

then the inequality (18) follows. $\square$

If n is an odd integer, then for any $\theta \in [0, 1]$ we have

$$\begin{aligned} (19) \quad & \left| \int_a^b f(x) dx - \frac{b-a}{2} \left[\theta f(a) + 2(1-\theta)f \left(\frac{a+b}{2} \right) + \theta f(b) \right] \right. \\ & \quad \left. - \sum_{k=1}^{n-1} \frac{[(-1)^k + 1][1 - (k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}(x) \right| \\ & \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!} \sqrt{\frac{(2n-1) - (4n^2-1)\theta + (2n+1)n^2\theta^2}{4n^2-1}} \sqrt{\sigma(f^{(n)})}, \end{aligned}$$

which also has been proved in Theorem 8 of [3] as well as Theorem 7 of [7], simultaneously.

If n is an even integer, then for any $\theta \in [0, 1]$ we have

$$\begin{aligned} (20) \quad & \left| \int_a^b f(x) dx - \frac{b-a}{2} \left[\theta f(a) + 2(1-\theta)f \left(\frac{a+b}{2} \right) + \theta f(b) \right] \right. \\ & \quad \left. - \sum_{k=1}^{n-1} \frac{[(-1)^k + 1][1 - (k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}(x) \right. \\ & \quad \left. - \frac{[1 - (n+1)\theta](b-a)^{n+1}}{(n+1)!2^n} D_n \right| \\ & \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!} \sqrt{\sigma(f^{(n)})} \times \\ & \quad \times \sqrt{\frac{2n^3 - n^2 - (4n^4 - 5n^2 + 1)\theta + (2n^5 + n^4 - 4n^3 - 2n^2 + 2n + 1)\theta^2}{4n^2 - 1}}. \end{aligned}$$

which also has been proved in Theorem 9 of [3] as well as Theorem 8 of [7], simultaneously.

Remark 3. If we take $\theta = 0$ in (19) and (20) then we get sharp midpoint type inequalities

$$(21) \quad \left| \int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n n! \sqrt{2n+1}} \sqrt{\sigma(f^{(n)})}$$

for an odd n and

$$(22) \quad \left| \int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \right. \\ \left. - \frac{(b-a)^n}{(n+1)!2^n} [f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\ \leq \frac{n(b-a)^{n+\frac{1}{2}}}{2^n (n+1)! \sqrt{2n+1}} \sqrt{\sigma(f^{(n)})}$$

for an even n .

In particular, for $n = 1$ in (21), we have

$$\left| \int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} \sqrt{\sigma(f')},$$

and for $n = 2$ in (22), we have

$$\left| \int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}} \sqrt{\sigma(f'')}.$$

Remark 4. If we take $\theta = 1$ in (19) and (20) then we get sharp trapezoid type inequalities

$$(23) \quad \left| \int_a^b f(x)dx - \frac{(b-a)}{2} [f(a) + f(b)] - \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!} \sqrt{\frac{2n^3 - 3n^2 + 2n}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}$$

for an odd n and

$$(24) \left| \int_a^b f(x)dx - \frac{(b-a)}{2}[f(a) + f(b)] - \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right. \\ \left. + \frac{n(b-a)^n}{(n+1)!2^n} [f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\ \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!} \sqrt{\frac{2n^5 - 3n^4 - 2n^3 + 2n^2 + 2n}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}$$

for an even n .

In particular, for $n = 1$ in (23), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)] \right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} \sqrt{\sigma(f')},$$

and for $n = 2$ in (24), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}} \sqrt{\sigma(f'')}.$$

Remark 5. If we take $\theta = \frac{1}{3}$ then we get sharp Simpson type inequalities

$$(25) \left| \int_a^b f(x)dx - \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right. \\ \left. + \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ \leq \frac{1}{3} \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!} \sqrt{\frac{2n^3 - 11n^2 + 18n - 6}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}$$

for an odd n and

$$(26) \left| \int_a^b f(x)dx - \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right. \\ \left. + \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right. \\ \left. + \frac{(n-2)(b-a)^n}{3(n+1)!2^n} [f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\ \leq \frac{1}{3} \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!} \sqrt{\frac{2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}$$

for an even n , which have been first proved in Theorem 5 and Theorem 6 of [9] in a more direct way.

In particular, for $n = 1, 3$ in (25) and $n = 2$ in (26), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{6} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] \right| \leq C_n(b-a)^{n+\frac{1}{2}} \sqrt{\sigma(f^{(n)})},$$

where $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{12\sqrt{30}}$, $C_3 = \frac{1}{48\sqrt{105}}$.

Remark 6. If we take $\theta = \frac{1}{2}$ then we have sharp averaged midpoint-trapezoid inequalities

$$(27) \quad \left| \int_a^b f(x)dx - \frac{b-a}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^{n+1}n!} \sqrt{\frac{2n^3 - 7n^2 + 8n - 2}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}$$

for an odd n and

$$(28) \quad \left| \int_a^b f(x)dx - \frac{b-a}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right) + \frac{(n-1)(b-a)^n}{(n+1)!2^{n+1}} [f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^{n+1}(n+1)!} \sqrt{\frac{2n^5 - 7n^4 + 4n^3 + 4n^2 + 2n - 1}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}$$

for an even n .

In particular, for $n = 1$ in (27), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}} \sqrt{\sigma(f')},$$

and for $n = 2$, in (28), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] + \frac{(b-a)^2}{48} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{48\sqrt{5}} \sqrt{\sigma(f'')},$$

Remark 7. If we take $\theta = \frac{7}{15}$ then we have sharp corrected Simpson type inequalities

$$(29) \quad \left| \int_a^b f(x)dx - \frac{b-a}{30} \left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b) \right] - \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(1-14k)(b-a)^{2k+1}}{15(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{n!2^n} \sqrt{\frac{98n^3 - 371n^2 + 450n - 120}{225(4n^2 - 1)}} \sqrt{\sigma(f^{(n)})}$$

for an odd n and

$$(30) \quad \left| \int_a^b f(x)dx - \frac{b-a}{30} \left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b) \right] - \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(1-14k)(b-a)^{2k+1}}{15(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) + \frac{(7n-8)(b-a)^n}{15(n+1)!2^{n+1}} [f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!} \sqrt{\frac{98n^5 - 371n^4 + 254n^3 + 202n^2 + 98n - 56}{225(4n^2 - 1)}} \sqrt{\sigma(f^{(n)})}$$

for an even n .

In particular, for $n = 1$ in (29), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{30} \left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b) \right] \right| \leq \frac{\sqrt{19}(b-a)^{\frac{3}{2}}}{30} \sqrt{\sigma(f')},$$

and for $n = 2$, in (30), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{30} \left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b) \right] + \frac{(b-a)^2}{60} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{60\sqrt{3}} \sqrt{\sigma(f'')},$$

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