

INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES ARE STRONGLY α -PREINVEX

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ABSTRACT. In the paper, the authors introduce a new notion “strongly α -preinvex function”, establish an integral identity for the newly introduced function, and find some Hermite–Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is strongly α -preinvex.

1. INTRODUCTION

Let us recall some definitions of various convex functions.

Definition 1.1. A function $f: I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$ is said to be convex if

$$(1.1) \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([1, 2, 14]). A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta: S \times S \rightarrow \mathbb{R}^n$ if for every $x, y \in S$ and $t \in [0, 1]$

$$(1.2) \quad y + t\eta(x, y) \in S.$$

Definition 1.3 ([8]). Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta: S \times S \rightarrow \mathbb{R}^n$. For every $x, y \in S$, the η -path P_{xy} joining the points x and $y = x + \eta(y, x)$ is defined by

$$(1.3) \quad P_{xy} = \{z \mid z = x + t\eta(y, x), t \in [0, 1]\}.$$

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Definition 1.4 ([2]). Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta: S \times S \rightarrow \mathbb{R}^n$. A function $f: S \rightarrow \mathbb{R}$ is said to be preinvex with respect to η , if for every $x, y \in S$ and $t \in [0, 1]$,

$$(1.4) \quad f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y).$$

Definition 1.5 ([10]). For $f: [a, b] \rightarrow \mathbb{R}$, if

$$(1.5) \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - c\lambda(1 - \lambda)(x - y)^2$$

is valid for all $x, y \in [a, b]$ and $\lambda \in [0, 1], c > 0$, then we say that $f(x)$ is a strongly convex function on $[a, b]$.

Let us reformulate some inequalities of Hermite–Hadamard type for the above mentioned convex functions.

Theorem 1.1 ([5, Theorem 2.2]). Let $f: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then

$$(1.6) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{b - a}{8} (|f'(a)| + |f'(b)|).$$

Theorem 1.2 ([2, Theorem 2.1]). Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta: A \times A \rightarrow \mathbb{R}$ and let $f: A \rightarrow \mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex on A , then for every $a, b \in A$ with $\theta(a, b) \neq 0$ we have

$$(1.7) \quad \left| \frac{f(b) + f(b + \theta(a, b))}{2} - \frac{1}{\theta(a, b)} \int_b^{b + \theta(a, b)} f(x) dx \right| \leq \frac{|\theta(a, b)|}{8} [|f'(a)| + |f'(b)|].$$

For more information on Hermite–Hadamard type inequalities for various convex functions, please refer to recently published articles [3, 4, 6, 7, 11, 12, 13] and closely related references therein.

In this article, we will introduce a new notion “ α -preinvex function”, establish an integral identity for such a kind of functions, and find some Hermite–Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is α -preinvex.

2. A NEW DEFINITION AND TWO LEMMAS

The so-called “strongly α -preinvex function” may be introduced as follows.

Definition 2.1. Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta: S \times S \rightarrow \mathbb{R}^n$. A function $f: S \rightarrow \mathbb{R}$ is said to be strongly α -preinvex with respect to η for $\alpha \in (0, 1]$ and $c > 0$, if for every $x, y \in S$ and $t \in [0, 1]$,

$$(2.1) \quad f(y + t\eta(x, y)) \leq t^\alpha f(x) + (1 - t^\alpha)f(y) - ct(1 - t)(x - y)^2.$$

For establishing our new integral inequalities of Hermite–Hadamard type for strongly α -preinvex functions, we need the following integral identity.

Lemma 2.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta: A \times A \rightarrow \mathbb{R}$ and let $a, b \in A$ with $\theta(a, b) \neq 0$. If $f: A \rightarrow \mathbb{R}$ is a differentiable function and f' is integrable on the θ -path P_{bc} , $c = b + \theta(a, b)$, then*

$$\begin{aligned} & \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx - f(b + \theta(a, b)) \\ &= \frac{\theta(a, b)}{4} \int_0^1 \left[(1+t)f' \left(b + \frac{1+t}{2} \theta(a, b) \right) + tf' \left(b + \frac{t}{2} \theta(a, b) \right) \right] dt. \end{aligned}$$

Proof. Since $a, b \in A$ and A is an invex set with respect to θ , for every $t \in [0, 1]$, we have $b + t\theta(a, b) \in A$. Integrating by part gives

$$\begin{aligned} & \int_0^1 \left[(1+t)f' \left(b + \frac{1+t}{2} \theta(a, b) \right) + tf' \left(b + \frac{t}{2} \theta(a, b) \right) \right] dt \\ &= \frac{2}{\theta(a, b)} \left[-(1+t)f \left(b + \frac{1+t}{2} \theta(a, b) \right) \Big|_0^1 + \int_0^1 f \left(b + \frac{1+t}{2} \theta(a, b) \right) dt \right. \\ & \quad \left. - tf \left(b + \frac{t}{2} \theta(a, b) \right) \Big|_0^1 + \int_0^1 f \left(b + \frac{t}{2} \theta(a, b) \right) dt \right] \\ &= \frac{2}{\theta(a, b)} \left[-2f(b + \theta(a, b)) + f \left(\frac{2b + \theta(a, b)}{2} \right) \right. \\ & \quad \left. + \frac{2}{\theta(a, b)} \int_{\frac{2b + \theta(a, b)}{2}}^{b + \theta(a, b)} f(x) \, dx - f \left(\frac{2b + \theta(a, b)}{2} \right) + \frac{2}{\theta(a, b)} \int_b^{[2b + \theta(a, b)]/2} f(x) \, dx \right] \\ &= \frac{4}{\theta(a, b)} \left[\frac{1}{\theta(a, b)} \int_b^{b + \theta(a, b)} f(x) \, dx - f(b + \theta(a, b)) \right]. \end{aligned}$$

The proof of Lemma 2.1 is completed. $\square$

By the same way, we obtain another lemma.

Lemma 2.2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta: A \times A \rightarrow \mathbb{R}$ and let $a, b \in A$ with $\theta(a, b) \neq 0$. If $f: A \rightarrow \mathbb{R}$ is a differentiable function and f' is integrable on the θ -path P_{bc} , $c = b + \theta(a, b)$, then*

$$\begin{aligned} & f(b) - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx \\ &= \frac{\theta(a, b)}{4} \int_0^1 \left[(1-t)f' \left(b + \frac{1+t}{2} \theta(a, b) \right) + (2-t)f' \left(b + \frac{t}{2} \theta(a, b) \right) \right] dt. \end{aligned}$$

3. SOME NEW INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE

We are now in a position to establish some Hermite–Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is strongly α -preinvex.

Theorem 3.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta: A \times A \rightarrow \mathbb{R}$, $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose $f: A \rightarrow \mathbb{R}$ be a differentiable function and f' is integrable on the θ -path P_{bc} , $c = b + \theta(a, b)$, some $\alpha \in (0, 1]$. If $|f'|^q$ is strongly α -preinvex on A for $q \geq 1$, then*

$$(3.1) \quad \left| \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) dx - f(b + \theta(a, b)) \right| \\ \leq \frac{|\theta(a, b)|}{8} \left(\frac{1}{(\alpha + 2)2^{\alpha-1}} \right)^{1/q} \left\{ 3[3(2^{\alpha+2} - 1)|f'(a)|^q \right. \\ \left. + (9 \times 2^{\alpha-1}(\alpha + 2) - 3(2^{\alpha+2} - 1))|f'(b)|^q - 11 \times 2^{\alpha-4}(\alpha + 2)c(a - b)^2 \right]^{1/q} \\ \left. + [3|f'(a)|^q + 3(2^{\alpha-1}(\alpha + 2) - 1)|f'(b)|^q - 5 \times 2^{\alpha-4}(\alpha + 2)c(a - b)^2]^{1/q} \right\}.$$

Proof. Since $b + t\theta(a, b) \in A$ for every $t \in [0, 1]$, by Lemma 2.1 and Hölder's inequality, we have

$$\left| \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) dx - f(b + \theta(a, b)) \right| \\ \leq \frac{|\theta(a, b)|}{4} \int_0^1 \left[(1+t) \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right| + t \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right| \right] dt \\ \leq \frac{|\theta(a, b)|}{4} \left\{ \left(\int_0^1 (1+t) dt \right)^{1-1/q} \left[\int_0^1 (1+t) \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right|^q dt \right]^{1/q} \right. \\ \left. + \left(\int_0^1 t dt \right)^{1-1/q} \left[\int_0^1 t \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right|^q dt \right]^{1/q} \right\} \\ \leq \frac{|\theta(a, b)|}{4} \left\{ \left(\frac{3}{2} \right)^{1-1/q} \left[\int_0^1 (1+t) \left(\left(\frac{1+t}{2} \right)^\alpha |f'(a)|^q \right. \right. \right. \\ \left. \left. + \left(1 - \left(\frac{1+t}{2} \right)^\alpha \right) |f'(b)|^q - \frac{1-t^2}{4} c(a-b)^2 \right) dt \right]^{1/q} \right. \\ \left. + \left(\frac{1}{2} \right)^{1-1/q} \left[\int_0^1 t \left(\left(\frac{t}{2} \right)^\alpha |f'(a)|^q \right. \right. \right. \\ \left. \left. + \left(1 - \left(\frac{t}{2} \right)^\alpha \right) |f'(b)|^q - \frac{2t-t^2}{4} c(a-b)^2 \right) dt \right]^{1/q} \right\} \\ = \frac{|\theta(a, b)|}{8} \left(\frac{1}{(\alpha + 2)2^{\alpha-1}} \right)^{1/q} \left\{ 3^{1-\frac{1}{q}} [(2^{\alpha+2} - 1)|f'(a)|^q \right. \\ \left. + (3 \times 2^{\alpha-1}(\alpha + 2) - (2^{\alpha+2} - 1))|f'(b)|^q - 11 \times 3^{-1} \times 2^{\alpha-4}(\alpha + 2)c(a - b)^2]^{1/q} \right. \\ \left. + [|f'(a)|^q + (2^{\alpha-1}(\alpha + 2) - 1)|f'(b)|^q - 5 \times 3^{-1} \times 2^{\alpha-4}(\alpha + 2)c(a - b)^2]^{1/q} \right\}.$$

The proof of Theorem 3.1 is completed. $\square$

Corollary 1. *Under the conditions of Theorem 3.1, if $\alpha = q = 1$, we have*

$$\begin{aligned} \left| \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx - f(b + \theta(a, b)) \right| \\ \leq \frac{|\theta(a, b)|}{12} [4|f'(a)|^q + 2|f'(b)|^q - c(a - b)^2]. \end{aligned}$$

Theorem 3.2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta: A \times A \rightarrow \mathbb{R}$, $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose $f: A \rightarrow \mathbb{R}$ be a differentiable function and f' is integrable on the θ -path P_{bc} , $c = b + \theta(a, b)$, some $\alpha \in (0, 1]$. If $|f'|^q$ is strongly α -preinvex on A for $q > 1$, then*

$$\begin{aligned} (3.2) \quad & \left| \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx - f(b + \theta(a, b)) \right| \\ & \leq \frac{|\theta(a, b)|}{4} \left(\frac{q-1}{(2q-1)(\alpha+1)2^\alpha} \right)^{1-1/q} \left(2^{\frac{2q-1}{q-1}} - 1 \right)^{1-1/q} \\ & \quad \times \left\{ \left[(2^\alpha - 1)|f'(a)|^q + (2^\alpha \alpha + 1)|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha+1)c(a-b)^2 \right]^{1/q} \right. \\ & \quad \left. + \left[|f'(a)|^q + (2^\alpha(\alpha+1) - 1)|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha+1)c(a-b)^2 \right]^{1/q} \right\}. \end{aligned}$$

Proof. Since $b + t\theta(a, b) \in A$ for every $t \in [0, 1]$, by Lemma 2.1, Hölder's inequality, we have

$$\begin{aligned} (3.3) \quad & \left| \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx - f(b + \theta(a, b)) \right| \\ & \leq \frac{|\theta(a, b)|}{4} \int_0^1 \left[(1+t) \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right| + t \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right| \right] \, dt \\ & \leq \frac{|\theta(a, b)|}{4} \left\{ \left(\int_0^1 (1+t)^{q/(q-1)} \, dt \right)^{1-1/q} \left[\int_0^1 \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right|^q \, dt \right]^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 t^{q/(q-1)} \, dt \right)^{1-1/q} \left[\int_0^1 \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right|^q \, dt \right]^{1/q} \right\} \\ & \leq \frac{|\theta(a, b)|}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left\{ \left(2^{\frac{2q-1}{q-1}} - 1 \right)^{1-1/q} \left[\int_0^1 \left(\left(\frac{1+t}{2} \right)^\alpha |f'(a)|^q \right. \right. \right. \\ & \quad \left. \left. + \left(1 - \left(\frac{1+t}{2} \right)^\alpha \right) |f'(b)|^q - \frac{1-t^2}{4} c(a-b)^2 \right) \, dt \right]^{1/q} \right. \\ & \quad \left. + \left[\int_0^1 t \left(\left(\frac{t}{2} \right)^\alpha |f'(a)|^q + \left(1 - \left(\frac{t}{2} \right)^\alpha \right) |f'(b)|^q - \frac{2t-t^2}{4} c(a-b)^2 \right) \, dt \right]^{1/q} \right\} \\ & = \frac{|\theta(a, b)|}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(\frac{1}{(\alpha+1)2^\alpha} \right)^{1-1/q} \left\{ \left(2^{\frac{2q-1}{q-1}} - 1 \right)^{1-1/q} \right. \end{aligned}$$

$$\begin{aligned} & \times \left[(2^\alpha - 1)|f'(a)|^q + (2^\alpha \alpha + 1)|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha + 1)c(a - b)^2 \right]^{1/q} \\ & + \left[|f'(a)|^q + (2^\alpha(\alpha + 1) - 1)|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha + 1)c(a - b)^2 \right]^{1/q} \Big\}. \end{aligned}$$

The proof of Theorem 3.2 is complete. $\square$

Corollary 2. *Under the conditions of Theorem 3.2, if $\alpha = 1$, we have*

$$\begin{aligned} & \left| \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx - f(b + \theta(a, b)) \right| \\ & \leq \frac{|\theta(a, b)|}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(\frac{1}{4} \right)^{1-1/q} \left\{ \left(2^{\frac{2q-1}{q-1}} - 1 \right)^{1-1/q} \right. \\ & \quad \times \left[|f'(a)|^q + 3|f'(b)|^q - 3^{-1} \times 2c(a - b)^2 \right]^{1/q} \\ & \quad \left. + \left[|f'(a)|^q + 3|f'(b)|^q - 3^{-1} \times 2c(a - b)^2 \right]^{1/q} \right\}. \end{aligned}$$

Theorem 3.3. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta: A \times A \rightarrow \mathbb{R}$, $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose $f: A \rightarrow \mathbb{R}$ be a differentiable function and f' is integrable on the θ -path P_{bc} , $c = b + \theta(a, b)$, some $\alpha \in (0, 1]$. If $|f'|^q$ is strongly α -preinvex on A for $q \geq 1$, then*

$$\begin{aligned} (3.4) \quad & \left| f(b) - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx \right| \\ & \leq \frac{|\theta(a, b)|}{8} \left(\frac{1}{2^{\alpha-1}(\alpha+1)(\alpha+2)} \right)^{1/q} \left\{ [(2^{\alpha+3} - 2^{\alpha+2} - \alpha - 3)|f'(a)|^q \right. \\ & \quad + [2^{\alpha-1}(\alpha+1)(\alpha+2) - (2^{\alpha+3} - 2^{\alpha+2} - \alpha - 3)]|f'(b)|^q \\ & \quad - 5 \times 2^{\alpha-4} \times 3^{-1}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q} + 3^{1-1/q}[(\alpha+3)|f'(a)|^q \\ & \quad + (3 \times 2^{\alpha-1}(\alpha+1)(\alpha+2) - \alpha - 3)|f'(b)|^q \\ & \quad \left. - 11 \times 3^{-1} \times 2^{\alpha-4}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q} \right\}. \end{aligned}$$

Proof. Since $b + t\theta(a, b) \in A$ for every $t \in [0, 1]$, by Lemma 2.2, Hölder's inequality, we have

$$\begin{aligned} & \left| f(b) - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) \, dx \right| \\ & \leq \frac{|\theta(a, b)|}{4} \int_0^1 \left[(1-t) \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right| + (2-t) \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right| \right] \, dt \\ & \leq \frac{|\theta(a, b)|}{4} \left\{ \left(\int_0^1 (1-t) \, dt \right)^{1-1/q} \left[\int_0^1 (1-t) \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right|^q \, dt \right]^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 (2-t) \, dt \right)^{1-1/q} \left[\int_0^1 (2-t) \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right|^q \, dt \right]^{1/q} \right\} \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{|\theta(a, b)|}{4} \left\{ \left(\frac{1}{2} \right)^{1-1/q} \left[\int_0^1 (1-t) \left(\left(\frac{1+t}{2} \right)^\alpha |f'(a)|^q + \left(1 - \left(\frac{1+t}{2} \right)^\alpha \right) |f'(b)|^q \right. \right. \right. \\
 &\quad \left. \left. - \frac{1-t^2}{4} c(a-b)^2 \right) dt \right]^{1/q} + \left(\frac{3}{2} \right)^{1-1/q} \left[\int_0^1 (2-t) \left(\left(\frac{t}{2} \right)^\alpha |f'(a)|^q \right. \right. \\
 &\quad \left. \left. + \left(1 - \left(\frac{t}{2} \right)^\alpha \right) |f'(b)|^q - \frac{2t-t^2}{4} c(a-b)^2 \right) dt \right]^{1/q} \right\} \\
 &= \frac{|\theta(a, b)|}{8} \left(\frac{1}{2^{\alpha-1}(\alpha+1)(\alpha+2)} \right)^{1/q} \left\{ [(2^{\alpha+3} - 2^{\alpha+2} - \alpha - 3)|f'(a)|^q \right. \\
 &\quad + [2^{\alpha-1}(\alpha+1)(\alpha+2) - (2^{\alpha+3} - 2^{\alpha+2} - \alpha - 3)]|f'(b)|^q \\
 &\quad - 5 \times 2^{\alpha-4} \times 3^{-1}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q} + 3^{1-1/q}[(\alpha+3)|f'(a)|^q \\
 &\quad + (3 \times 2^{\alpha-1}(\alpha+1)(\alpha+2) - \alpha - 3)|f'(b)|^q \\
 &\quad \left. - 11 \times 3^{-1} \times 2^{\alpha-4}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q} \right\}.
 \end{aligned}$$

The proof of Theorem 3.3 is complete. $\square$

Corollary 3. Under the conditions of Theorem 3.3, if $\alpha = q = 1$, we have

$$\left| f(b) - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) dx \right| \leq \frac{|\theta(a, b)|}{24} [4|f'(a)|^q + 7|f'(b)|^q - 2c(a-b)^2].$$

Theorem 3.4. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta: A \times A \rightarrow \mathbb{R}$, $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose $f: A \rightarrow \mathbb{R}$ be a differentiable function and f' is integrable on the θ -path P_{bc} , $c = b + \theta(a, b)$, some $\alpha \in (0, 1]$. If $|f'|^q$ is strongly α -preinvex on A for $q > 1$, then

$$\begin{aligned}
 (3.5) \quad &\left| f(b) - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) dx \right| \\
 &\leq \frac{|\theta(a, b)|}{8} \left(\frac{1}{2^{\alpha-1}(\alpha+1)} \right)^{1/q} \left\{ [(2^{\alpha+1} - 1)|f'(a)|^q + (2^\alpha(\alpha+1) \right. \\
 &\quad \left. + 3^{1-1/q}[|f'(a)|^q - (2^{\alpha+1} - 1)]|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q} \right. \\
 &\quad \left. + (2^\alpha(\alpha+1) - 1)|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q} \right\}.
 \end{aligned}$$

Proof. Since $b + t\theta(a, b) \in A$ for every $t \in [0, 1]$, by Lemma 2.2, Hölder's inequality, we have

$$\begin{aligned}
 &\left| f(b) - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) dx \right| \\
 &\leq \frac{|\theta(a, b)|}{4} \int_0^1 \left[(1-t) \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right| + (2-t) \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right| \right] dt \\
 &\leq \frac{|\theta(a, b)|}{4} \left\{ \left(\int_0^1 (1-t)^{q/(q-1)} dt \right)^{1-1/q} \left[\int_0^1 \left| f' \left(b + \frac{1+t}{2} \theta(a, b) \right) \right|^q dt \right]^{1/q} \right. \\
 &\quad \left. + \left(\int_0^1 (2-t)^{q/(q-1)} dt \right)^{1-1/q} \left[\int_0^1 \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right|^q dt \right]^{1/q} \right\}.
 \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 (2-t)^{q/(q-1)} dt \right)^{1-1/q} \left[\int_0^1 \left| f' \left(b + \frac{t}{2} \theta(a, b) \right) \right|^q dt \right]^{1/q} \Big\} \\
& \leq \frac{|\theta(a, b)|}{4} \left\{ \left(\frac{1}{2} \right)^{1-1/q} \left[\int_0^1 \left(\left(\frac{1+t}{2} \right)^\alpha |f'(a)|^q + \left(1 - \left(\frac{1+t}{2} \right)^\alpha \right) |f'(b)|^q \right. \right. \right. \\
& \quad \left. \left. - \frac{1-t^2}{4} c(a-b)^2 \right) dt \right]^{1/q} + \left(\frac{3}{2} \right)^{1-1/q} \left[\int_0^1 \left(\left(\frac{t}{2} \right)^\alpha |f'(a)|^q \right. \right. \\
& \quad \left. \left. + \left(1 - \left(\frac{t}{2} \right)^\alpha \right) |f'(b)|^q - \frac{2t-t^2}{4} c(a-b)^2 \right) dt \right]^{1/q} \right\} \\
& = \frac{|\theta(a, b)|}{8} \left(\frac{1}{2^{\alpha-1}(\alpha+1)} \right)^{1/q} \left\{ [(2^{\alpha+1}-1)|f'(a)|^q + (2^\alpha(\alpha+1) \right. \\
& \quad \left. - (2^{\alpha+1}-1))|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q} + 3^{1-1/q} [|f'(a)|^q \right. \\
& \quad \left. + (2^\alpha(\alpha+1)-1)|f'(b)|^q - 3^{-1} \times 2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q} \right\}.
\end{aligned}$$

The proof of Theorem 3.4 is complete. $\square$

Corollary 4. *Under the conditions of Theorem 3.4, if $\alpha = 1$, we have*

$$\begin{aligned}
& \left| f(b) - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a, b)} f(x) dx \right| \\
& \leq \frac{|\theta(a, b)|}{8} \left(\frac{1}{2} \right)^{1/q} \left\{ [3|f'(a)|^q + 3|f'(b)|^q - 3^{-1} \times 2c(a-b)^2]^{1/q} \right. \\
& \quad \left. + 3^{1-1/q} [|f'(a)|^q + 3|f'(b)|^q - 3^{-1} \times 2c(a-b)^2]^{1/q} \right\}.
\end{aligned}$$

Remark. On 8 April 2014, Professor S. S. Dragomir, Australia pointed out that there are errors appeared in the paper [9] as follows. Definition 2.1 from [9] has no meaning if i stands for the imaginary unit. So, no inequalities like in the hypothesis of Theorem 3.1 or eq. (3.1) and others can be stated for general φ .

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