

A CLASS OF FINSLER METRICS WITH ISOTROPIC MEAN BERWALD CURVATURE

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ABSTRACT. In this paper, we find a condition on (α, β) -metrics under which the notions of isotropic S -curvature, weakly isotropic S -curvature and isotropic mean Berwald curvature are equivalent.

1. INTRODUCTION

The S -curvature is introduced by Shen for a comparison theorem on Finsler manifolds [8]. Recent studies show that the S -curvature plays a very important role in Finsler geometry [11, 12]. A Finsler metric F is said to have isotropic S -curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on an n -dimensional manifold M .

Taking twice vertical covariant derivatives of the S -curvature gives rise the mean Berwald curvature. A Finsler metric F with vanishing mean Berwald curvature is called weakly Berwald metric. In [1], Bácsó and Yoshikawa studied some weakly Berwald metrics. Also, F is called to have isotropic mean Berwald curvature if $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$, for some scalar function c on M , where $\mathbf{h}$ is the angular metric. It is easy to see that every Finsler metric of isotropic S -curvature is of isotropic mean Berwald curvature. Now, is the equation $\mathbf{S} = (n+1)cF$ equivalent to the equation $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$?

Recently, Cheng and Shen proved that a Randers metric $F = \alpha + \beta$ is of isotropic S -curvature if and only if it is of isotropic mean Berwald curvature [2]. Then Xiang and Cheng extended this equivalency to the Finsler metric $F = \alpha^{-m}(\alpha + \beta)^{m+1}$ for every real constant m , including Randers metric [13]. In [7] Lee and Lee proved that this notions are equivalent for the Finsler metrics in the form $F = \alpha + \alpha^{-1}\beta^2$.

All of above metrics are special Finsler metrics so-called (α, β) -metrics. An (α, β) -metric is a scalar function on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$ where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on a manifold M . A natural question arises:

2010 *Mathematics Subject Classification.* 53B40, 53C60.

Key words and phrases. (α, β) -metric, isotropic S -curvature, isotropic E -curvature.

Is being of isotropic S -curvature equivalent to being of isotropic mean Berwald curvature for (α, β) -metrics?

In [6] Deng and Wang found the formula of the S -curvature of homogeneous (α, β) -metrics. Then Cheng and Shen classified (α, β) -metrics of isotropic S -curvature [3].

Let $F = \alpha\phi(s)$ be an (α, β) -metric on a manifold M of dimension n , where $s = \frac{\beta}{\alpha}$, $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . For an (α, β) -metric, put

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Delta &= 1 + sQ + (b^2 - s^2)Q', \\ \Phi &= -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'', \\ \Xi &= \frac{(b^2Q + s)\Phi}{\Delta^2}. \end{aligned}$$

Using the same method as in [3], we give an affirmative answer to the above question for almost all (α, β) -metrics. More precisely, we prove the following.

Theorem 1.1. *Let $F = \alpha\phi(s)$ be an (α, β) -metric, where $s = \frac{\beta}{\alpha}$. Suppose that Ξ is not constant. Then F is of isotropic S -curvature if and only if it is of isotropic mean Berwald curvature.*

It is remarkable that if $\Xi = 0$, then F reduces to a Riemannian metric. But, in general, it is still an open problem if Theorem 1.1 is true when Ξ is a constant.

Example 1.2. The above mentioned (α, β) -metric correspond to $\phi = 1 + s$, $\phi = (1 + s)^{m+1}$ and $\phi = 1 + s^2$, respectively. Using a Maple program shows that for all these metrics Ξ is not constant.

2. PRELIMINARIES

Let $F = F(x, y)$ be a Finsler metric on an n -dimensional manifold M . There is a notion of distortion $\tau = \tau(x, y)$ on TM associated with a volume form $dV = \sigma(x)dx$, which is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S -curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt} \left[\tau(c(t), \dot{c}(t)) \right] \Big|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ [5, 10]. From the definition, we see that the S -curvature $\mathbf{S}(x, y)$ measures the rate of change in the distortion on $(T_x M, F_x)$ in the direction $y \in T_x M$.

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ denote the spray of F and $dV_{BH} = \sigma(x)dx$ be the Busemann-Hausdorff volume form on M , where the spray coefficients G^i are defined by

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y) \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l}(x, y) y^k - \frac{\partial [F^2]}{\partial x^l}(x, y) \right\}, \quad y \in T_x M.$$

Then the S-curvature is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} (\ln \sigma).$$

The mean Berwald curvature $\mathbf{E} = E_{ij} dx^i \otimes dx^j$ is given by

$$E_{ij} = \frac{1}{2} \frac{\partial^2 S}{\partial y^i \partial y^j}.$$

Definition 2.1. Let (M, F) be an n -dimensional Finsler manifold. Then

- (a) F is of isotropic S -curvature if $\mathbf{S} = (n+1)cF$,
- (b) F is of weak isotropic S -curvature if $\mathbf{S} = (n+1)cF + \eta$,
- (c) F is of isotropic mean Berwald curvature if $\mathbf{E} = \frac{n+1}{2} cF^{-1} \mathbf{h}$,

where $c = c(x)$ is a scalar function on M , $\eta = \eta_i(x)y^i$ is a 1-form on M and $\mathbf{h}$ is the angular metric [9].

Consider the (α, β) -metric $F = \alpha \phi\left(\frac{\beta}{\alpha}\right)$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M . For an (α, β) -metric, put

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad r_{i0} := r_{ij} y^j, \quad s_{i0} := s_{ij} y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

Let $\bar{G}^i$ denote the spray coefficients of α . We have the following formula for the spray coefficients G^i of F [5]:

$$G^i = \bar{G}^i + \alpha Q s^i_0 + \Theta \left\{ -2Q\alpha s_0 + r_{00} \right\} \frac{y^i}{\alpha} + \Psi \left\{ -2Q\alpha s_0 + r_{00} \right\} b^i,$$

where $s^i_j := a^{ih} s_{hj}$, $s^i_0 := s^i_j y^j$ and $r_{00} := r_{ij} y^i y^j$. In [3], Cheng-Shen found the S -curvature as follows

$$(1) \quad \mathbf{S} = \left\{ 2\Psi - \frac{f'(b)}{bf(b)} \right\} (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q s_0),$$

where

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Delta = 1 + sQ + (b^2 - s^2)Q', \quad \Psi = \frac{Q'}{2\Delta}$$

$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''.$$

Recently, Cheng and Shen characterized (α, β) -metrics with isotropic S -curvature and proved the following.

Lemma 2.2 ([3]). *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric on an n -manifold. Then, F is of isotropic S -curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) β satisfies

$$(2) \quad r_{ij} = \varepsilon \left\{ b^2 a_{ij} - b_i b_j \right\}, \quad s_j = 0,$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$(3) \quad \Phi = -2(n+1)k \frac{\phi \Delta^2}{b^2 - s^2},$$

where k is a constant. In this case, $c = k\varepsilon$.

(ii) β satisfies

$$(4) \quad r_{ij} = 0, \quad s_j = 0.$$

In this case, $c = 0$.

It is remarkable that Cheng, Wang and Wang proved that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α, β) -metrics [4]. Hence, in the continue, we suppose that $\Phi \neq 0$.

3. PROOF OF THEOREM 1.1

First, we find the formula of mean Berwald curvature of (α, β) -metrics. After a long and tedious computation, we obtain the following.

Proposition 3.1. *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β) -metric. Put $\Omega := \frac{\Phi}{2\Delta^2}$. Then the mean Berwald curvature of F is given by the following*

$$(5) \quad E_{ij} = C_1 b_i b_j + C_2 (b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5 (r_{i0} b_j + r_{j0} b_i) \\ + C_6 (r_{i0} y_j + r_{j0} y_i) + C_7 r_{ij} + C_8 (s_i b_j + s_j b_i) + C_9 (s_i y_j + s_j y_i) \\ + C_{10} (r_i b_j + r_j b_i) + C_{11} (r_i y_j + r_j y_i),$$

where

$$C_1 := \frac{1}{2\alpha^3 \Delta^2} \left\{ \Phi \alpha Q'' s_0 + 2\alpha \Delta^2 \Psi'' r_0 - \Delta^2 \Omega'' r_0 + 2\Delta^2 \alpha \Omega'' Q s_0 \right. \\ \left. + 4\Delta^2 \alpha \Omega' Q' s_0 + 2\alpha \Delta^2 \Psi'' s_0 \right\},$$

$$C_2 := \frac{-1}{2\alpha^4 \Delta^2} \left\{ 2\alpha \Delta^2 \Psi'' s_0 - 2\Omega' \Delta^2 r_0 + 2\Omega' \Delta^2 \alpha Q s_0 - \Delta^2 \Omega'' s r_0 \right. \\ + 2\Delta^2 \alpha \Omega'' s Q s_0 + 4\Delta^2 \alpha \Omega' Q' s_0 s + 2\alpha \Delta^2 \Psi' r_0 + 2\alpha \Delta^2 \Psi'' s r_0 \\ \left. + 2\alpha \Delta^2 \Psi'' s s_0 + \Phi \alpha Q' s_0 + \Phi \alpha Q'' s_0 s \right\},$$

$$C_3 := \frac{1}{4\alpha^5 \Delta^2} \left\{ 4\Delta^2 s^2 \Omega'' \alpha Q s_0 - 2\Delta^2 s^2 \Omega'' r_0 + 12\alpha \Delta^2 \Psi' s r_0 + 12\alpha \Delta^2 \Psi' s s_0 \right. \\ + 4\alpha \Delta^2 \Psi'' s^2 r_0 + 4\alpha \Delta^2 \Psi'' s^2 s_0 + 8\Delta^2 s^2 \Omega' \alpha Q' s_0 + 2\Phi \alpha Q'' s_0 s^2 \\ \left. - 10\Omega' \Delta^2 s r_0 + 12\Omega' \Delta^2 s \alpha Q s_0 + 6\Phi \alpha Q' s_0 s - 3\Phi r_0 \right\},$$

$$\begin{aligned}
C_4 &:= \frac{-1}{4\alpha^3\Delta^2} \left\{ 4\alpha\Delta^2\Psi'ss_0 - \Phi r_0 - 2\Omega'\Delta^2sr_0 + 4\Omega'\Delta^2s\alpha Qs_0 \right. \\
&\quad \left. + 4\alpha\Delta^2\Psi'sr_0 + 2\Phi\alpha Q's_0s \right\}, \\
C_5 &:= \frac{-\Omega'}{\alpha^2}, \quad C_6 := \frac{2\Delta^2s\Omega' + \Phi}{2\alpha^3\Delta^2}, \quad C_7 := \frac{-\Phi}{2\alpha\Delta^2}, \\
C_8 &:= \frac{1}{2\alpha\Delta^2} \{ 2\Omega'\Delta^2Q + 2\Delta^2\Psi' + \Phi Q' \}, \\
C_9 &:= \frac{-s}{\alpha} C_8, \quad C_{10} := \frac{\Psi'}{\alpha}, \quad C_{11} := \frac{-s}{\alpha} C_{10}.
\end{aligned}$$

The formula of mean Berwald curvature of Randers metrics and Kropina metrics computed from Proposition 3.1 coincides with the one computed in [1].

It is easy to see that F is of isotropic mean Berwald curvature if and only if F is of weak isotropic S -curvature. Hence, we consider an (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ with weak isotropic S -curvature, $\mathbf{S} = (n+1)cF + \eta$, where $\eta = \eta_i(x)y^i$ is a 1-form on underlying manifold M . Using the same method used in [3], one can obtain that the condition that F is of weak isotropic S -curvature $\mathbf{S} = (n+1)cF + \eta$ is equivalent to the following equation

$$(6) \quad \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Qs_0) - 2\Psi(r_0 + s_0) = -(n+1)cF + \tilde{\theta},$$

where

$$(7) \quad \tilde{\theta} := -\frac{f'(b)}{bf(b)}(r_0 + s_0) - \eta.$$

To simplify the equation (6), we choose special coordinates $\psi: (s, u^A) \rightarrow (y^i)$ as follows

$$(8) \quad y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^A = u^A,$$

where

$$\bar{\alpha} = \sqrt{\sum_{A=2}^n (u^A)^2}.$$

Then

$$(9) \quad \alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$

Fix an arbitrary point x . Take a local coordinate system at x as in (8). We have

$$\begin{aligned}
r_1 &= br_{11}, \quad r_A = br_{1A}, \\
s_1 &= 0, \quad s_A = bs_{1A}.
\end{aligned}$$

Let

$$\begin{aligned}\bar{r}_{10} &:= \sum_{A=2}^n r_{1A} y^A, & \bar{s}_{10} &:= \sum_{A=2}^n s_{1A} y^A, & \bar{r}_{00} &:= \sum_{A,B=2}^n r_{AB} y^A y^B, \\ \bar{r}_0 &:= \sum_{A=2}^n r_A y^A, & \bar{s}_0 &:= \sum_{A=2}^n s_A y^A.\end{aligned}$$

Put

$$\tilde{\theta} = t_i y^i - \eta_i y^i.$$

Then t_i are given by

$$(10) \quad t_1 = -\frac{f'(b)}{f(b)} r_{11}, \quad t_A = -\frac{f'(b)}{f(b)} (r_{1A} + s_{1A}).$$

From (8), we have

$$(11) \quad r_0 = \frac{s b r_{11}}{\sqrt{b^2 - s^2}} \bar{\alpha} + b \bar{r}_{10}, \quad s_0 = \bar{s}_0 = b \bar{s}_{10},$$

and

$$(12) \quad r_{00} = \frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + 2 \frac{s \bar{\alpha}}{\sqrt{b^2 - s^2}} \bar{r}_{10} + \bar{r}_{00},$$

$$(13) \quad \tilde{\theta} = t_1 \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} - \frac{f'(b)}{f(b)} \bar{r}_{10} - \frac{f'(b)}{f(b)} \bar{s}_{10} - \eta.$$

Substituting (11), (12) and (13) into (6) and by using (9), we find that (6) is equivalent to the following equations:

$$(14) \quad \frac{\Phi}{2\Delta^2} (b^2 - s^2) \bar{r}_{00} = - \left\{ s \left(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 \right) r_{11} + (n+1) c b^2 \phi - s b t_1 \right\} \bar{\alpha}^2,$$

$$(15) \quad \left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) (r_{1A} + s_{1A}) - (b^2 Q + s) \frac{\Phi}{\Delta^2} s_{1A} + b \eta_A - b t_A = 0.$$

$$(16) \quad \eta_1 = 0.$$

Let

$$\Upsilon := \left[\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right]'$$

We see that $\Upsilon = 0$ if and only if

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2 \mu,$$

where $\mu = \mu(x)$ is independent of s .

Let us suppose that $\Xi = \frac{(b^2 Q + s)\Phi}{\Delta^2}$ is not constant. Now we shall divide the proof into two cases:

(i) $\Upsilon = 0$ and (ii) $\Upsilon \neq 0$.

3.1. $\Upsilon = 0$. First, note that $\Upsilon = 0$ implies that

$$(17) \quad \frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu,$$

where $\mu = \mu(x)$ is a function on M independent of s . First, we prove the following.

Lemma 3.2. *Let (M, F) be an n -dimensional Finsler manifold. Suppose that $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric and $\Upsilon = 0$. If F has weak isotropic S -curvature, $\mathbf{S} = (n+1)cF + \eta$, then β satisfies*

$$(18) \quad r_{ij} = ka_{ij} - \varepsilon b_i b_j + \frac{1}{b^2}(r_i b_j + r_j b_i),$$

where $k = k(x)$, $\varepsilon = \varepsilon(x)$, and $\phi = \phi(s)$ satisfies the following ODE:

$$(19) \quad (k - \varepsilon s^2) \frac{\Phi}{2\Delta^2} = \left\{ \nu + (k - \varepsilon b^2)\mu \right\} s - (n+1)c\phi,$$

where $\nu = \nu(x)$. If $s_0 \neq 0$, then ϕ satisfies the following additional ODE:

$$(20) \quad \frac{\Phi}{\Delta^2}(Qb^2 + s) = b^2(\mu + \lambda),$$

where $\lambda = \lambda(x)$.

Proof. Since $\Phi \neq 0$ and $\bar{r}_{00}$ and $\bar{\alpha}$ are independent of s , it follows from (14) and (15) that in a special coordinate system (s, y^a) at a point x , the following relations hold

$$(21) \quad r_{AB} = k\delta_{AB},$$

$$(22) \quad s \left(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 \right) r_{11} + (n+1)cb^2\phi + k \frac{\Phi}{2\Delta^2}(b^2 - s^2) = bst_1,$$

$$(23) \quad \left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) (r_{1A} + s_{1A}) - (b^2Q + s) \frac{\Phi}{\Delta^2} s_{1A} - bt_A = -b\eta_A,$$

where $k = k(x)$ is independent of s . Let

$$r_{11} = -(k - \varepsilon b^2).$$

Then (18) holds. By (17), we have

$$\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 = b^2\mu - \frac{s\Phi}{2\Delta^2}.$$

Then (22) and (23) become

$$(24) \quad b(k - \varepsilon s^2) \frac{\Phi}{2\Delta^2} = st_1 + sb\mu(k - b^2\varepsilon) - (n+1)cb\phi.$$

$$(25) \quad b^2\mu(r_{1A} + s_{1A}) - \frac{\Phi}{\Delta^2}(Qb^2 + s)s_{1A} - bt_A = -b\eta_A.$$

Letting $t_1 = b\nu$ in (24) we get (19). Now, suppose that $s_0 \neq 0$. Rewrite (25) as

$$\left\{ b^2\mu - \frac{\Phi}{\Delta^2}(Qb^2 + s) \right\} s_{1A} = bt_A - b\eta_A - b^2\mu r_{1A}.$$

We can see that there is a function $\lambda = \lambda(x)$ on M such that

$$\mu b^2 - \frac{\Phi}{\Delta^2}(Qb^2 + s) = -b^2\lambda.$$

This gives (20). □

Lemma 3.3 ([3]). *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric. Assume that*

$$\phi \neq k_1\sqrt{1 + k_2s^2} + k_3s$$

for any constants $k_1 > 0, k_2$ and k_3 . If $\Upsilon = 0$, then $b = \text{constant}$.

An (α, β) -metric is called Randers-type if $\phi = k_1\sqrt{1 + k_2s^2} + k_3s$ for any constants $k_1 > 0, k_2$ and k_3 . Now, we consider the equivalency of the notions weak isotropic S -curvature and isotropic S -curvature for a non-Randers type (α, β) -metric.

Lemma 3.4. *Let $F = \alpha\phi(\beta/\alpha)$ be a non-Randers type (α, β) -metric. Suppose that Ξ is not constant and $\Upsilon = 0$. Then F is of weak isotropic S -curvature if and only if F is of isotropic S -curvature.*

Proof. It is sufficient to prove that if F is of weak isotropic S -curvature, then F is of isotropic S -curvature. By $db = (r_0 + s_0)/b$ and Lemma 3.3, we have

$$r_0 + s_0 = 0.$$

Then by the formula of S -curvature of an (α, β) -curvature, we get

$$\mathbf{S} = -\alpha^{-1} \frac{\Phi}{2\Delta^2} \left\{ r_{00} - 2\alpha Q s_0 \right\}.$$

By Lemma 3.2,

$$r_{00} = (k - \varepsilon s^2)\alpha^2 + \frac{2s}{b^2}r_0\alpha.$$

Then

$$\mathbf{S} = -(k - \varepsilon s^2) \frac{\Phi}{2\Delta^2} \alpha + \frac{\Phi}{b^2\Delta^2} (b^2Q + s)s_0.$$

By (19), we have

$$(26) \quad \mathbf{S} = -s \left\{ \nu + (k - \varepsilon b^2)\mu \right\} \alpha + \frac{\Phi}{b^2\Delta^2} (b^2Q + s)s_0 + (n+1)c\phi\alpha.$$

Since $\mathbf{S} = (n+1)cF + \eta$, then by (26) we obtain the following

$$(27) \quad -s \left\{ \nu + (k - \varepsilon b^2)\mu \right\} \alpha + \frac{\Phi}{b^2\Delta^2} (b^2Q + s)s_0 = \eta.$$

Letting $y^i = \delta b^i$ for a sufficiently small $\delta > 0$ yields

$$-\delta \left\{ \nu + (k - \varepsilon b^2) \mu \right\} b^2 = \delta \eta_i b^i.$$

It is easy to see that in the special coordinate $\eta_i b^i = 0$, hence in general $\eta_i b^i = 0$. We conclude that

$$(28) \quad \nu + (k - \varepsilon b^2) \mu = 0.$$

Then (27) reduces to

$$(29) \quad \frac{\Xi}{b^2} s_0 = \eta.$$

If $s_0 \neq 0$, then from the last equation, we obtain that Ξ is constant, which is excluded here. Hence, we have $s_0 = 0$. Thus by (29), we conclude that $\eta = 0$ and F has isotropic S -curvature $\mathbf{S} = (n+1)cF$. $\square$

3.2. $\Upsilon \neq 0$. Here, we consider the case when $\phi = \phi(s)$ satisfies

$$(30) \quad \Upsilon \neq 0$$

We need the following two lemmas. The proofs mainly follow the proof of Lemma 6.1 and Lemma 6.2 in [3], respectively. Thus we omit the proofs.

Lemma 3.5. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on an n -dimensional manifold. Assume that $\Upsilon \neq 0$. Suppose that F has weak isotropic S -curvature, $\mathbf{S} = (n+1)cF + \eta$. Then*

$$(31) \quad r_{ij} = ka_{ij} - \varepsilon b_i b_j - \lambda(s_i b_j + s_j b_i),$$

where $\lambda = \lambda(x)$, $k = k(x)$ and $\varepsilon = \varepsilon(x)$ are scalar functions of x and

$$(32) \quad -2s(k - \varepsilon b^2)\Psi + (k - \varepsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu = 0,$$

where

$$(33) \quad \nu := -\frac{f'(b)}{bf(b)}(k - \varepsilon b^2).$$

If in addition $s_0 \neq 0$, i.e., $s_{A_o} \neq 0$ for some A_o , then

$$(34) \quad -2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda \left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) = \delta,$$

where

$$(35) \quad \delta := -\frac{f'(b)}{bf(b)}(1 - \lambda b^2) - \frac{\eta_{A_o}}{s_{A_o}}.$$

Lemma 3.6. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric. Suppose that $\phi = \phi(s)$ satisfies (30) and $\phi \neq k_1\sqrt{1+k_2s^2} + k_3s$ for any constants $k_1 > 0$, k_2 and k_3 . If F has weak isotropic S -curvature, then*

$$r_j + s_j = 0.$$

Proposition 3.7. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric. Suppose that $\phi = \phi(s)$ satisfies (30) and $\phi \neq k_1\sqrt{1 + k_2s^2} + k_3s$ for any constants $k_1 > 0$, k_2 and k_3 . Suppose that Ξ is not constant. If F is of weak isotropic S -curvature, $\mathbf{S} = (n + 1)cF + \eta$, then*

$$(36) \quad r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j), \quad s_j = 0,$$

where $\varepsilon = \varepsilon(x)$ is a scalar function on M and $\phi = \phi(s)$ satisfies

$$(37) \quad \varepsilon(b^2 - s^2)\frac{\Phi}{2\Delta^2} = -(n + 1)c\phi.$$

Proof. Contracting (31) with b^i yields

$$(38) \quad r_j + s_j = (k - \varepsilon b^2)b_j + (1 - \lambda b^2)s_j.$$

By Lemma 3.6, $r_j + s_j = 0$. It follows from (38) that

$$(39) \quad (1 - \lambda b^2)s_j + (k - \varepsilon b^2)b_j = 0.$$

Contracting (39) with b^j yields

$$(k - \varepsilon b^2)b^2 = 0.$$

We get

$$k = \varepsilon b^2.$$

Then (31) is reduced to

$$r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j) - \lambda(b_is_j + b_js_i).$$

By (33),

$$\nu = 0.$$

Then (32) is reduced to (37).

We claim that $s_0 = 0$. Suppose that $s_0 \neq 0$. By (39), we conclude that

$$\lambda = \frac{1}{b^2}.$$

By (35),

$$\delta = -\frac{\eta_{A_0}}{s_{A_0}}.$$

It follows from (34) that

$$\frac{(b^2Q + s)\Phi}{\Delta^2} = \frac{b\eta_{A_0}}{s_{A_0}},$$

which implies that Ξ is constant. This is impossible by the assumption on non-constancy of Ξ . Therefore, $s_j = 0$. This completes the proof. $\square$

By Proposition 3.7 and Lemma 2.2, we have the following.

Corollary 3.8. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α, β) -metric. Suppose that $\Upsilon \neq 0$ and Ξ is not constant. Then F is of weak isotropic S -curvature, if and only if it is of isotropic S -curvature.*

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Received March 5, 2015.

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