

ON A FINSLER SPACE WITH A SPECIAL METRICAL CONNECTION

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ABSTRACT. In this paper, we consider a Finsler space with a special metrical connection and find necessary and sufficient condition when the (v)hv-torsion tensor ${}^*P_{jk}^i$ with respect to the special metrical connection coincides with the (v)hv-torsion P_{jk}^i with respect to general Finsler connection. The relation in hv-curvature tensor, h-curvature tensor and v(h)-torsion tensor with respect to these two connection are also obtained.

1. INTRODUCTION

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with the fundamental function $L(x, y)$. The metric tensor, the angular metric tensor and Cartan tensor are defined by

$$g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \quad h_{ij} = L \dot{\partial}_i \dot{\partial}_j L \quad \text{and} \quad C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$$

respectively, where $\dot{\partial}_i = \partial / \partial y^i$.

A Finsler connection is a triad $F\Gamma = (\Gamma_{jk}^i, N_j^i, C_{jk}^i)$, where Γ_{jk}^i are connection coefficients of h-connection, N_j^i are connection coefficients of non-linear connection and C_{jk}^i are connection coefficients of v-connection. For a given connection, the h- and v-covariant derivatives of any vector X^i are given by

$$X^i|_k = \delta_k X^i + X^r \Gamma_{rk}^i,$$

and

$$X^i|_k = \dot{\partial}_k X^i + X^r C_{rk}^i,$$

where $\delta_k = \partial_k - N_k^r \dot{\partial}_r$ and $\partial_k = \frac{\partial}{\partial x^k}$.

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In 2006, H. S. Park et.al. [4] defined a new non-linear connection $\overline{N}_j^i$ with the help of given non-linear connection N_j^i for an (α, β) -metric as

$$(1.1) \quad \overline{N}_j^i = N_j^i + \nabla_j L \frac{y^i}{L},$$

where ' ∇_j ' denotes the covariant derivative with respect to the associated Riemannian connection.

In 2008, H. G. Nagaraja [3] defined a new non-linear connection $*N_j^i$ with the help of given non-linear connection for Randers space as

$$(1.2) \quad *N_j^i = N_j^i + \frac{L_{|j} y^i}{L},$$

where ' $_{|j}$ ' denote the covariant derivative with respect to Finsler connection $F\Gamma$, and find a new Finsler connection $*F\Gamma = (\Gamma_{jk}^i, *N_j^i, C_{jk}^i)$.

In this paper, we consider a Finsler space F^n admitting the Finsler connection $*F\Gamma$ and we find a relation between $v(hv)$ -torsion tensors with respect to these two Finsler connection connections $F\Gamma$ and $*F\Gamma$. We obtain necessary and sufficient condition that two $(v)hv$ -torsions coincides. We also find relation in hv -curvature tensor, h -curvature tensor and $v(h)$ torsion tensor with respect to these two Finsler connections.

The Terminology and notion are referred to [2, 5].

2. A SPECIAL METRICAL CONNECTION

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space and $F\Gamma = (\Gamma_{jk}^i, N_j^i, C_{jk}^i)$ be a Finsler connection. Let the Finsler space $F^n = (M^n, L)$ admits a new Finsler connection $*F\Gamma = (*\Gamma_{jk}^i, *N_j^i, C_{jk}^i)$, which is $h(h)$ -torsion free and non-linear coefficients $*N_j^i$ are given by (1.4). Then we have

$$(2.1) \quad *\delta_k = \partial_k - *N_k^r \dot{\partial}_r = \partial_k - N_k^r \dot{\partial}_r - \frac{L_{|k} y^r}{L} \dot{\partial}_r = \delta_k - \frac{L_{|k} y^r}{L} \dot{\partial}_r.$$

The h -covariant derivative of L with respect to $*F\Gamma$ is given by

$$(2.2) \quad L_{*|k} = *\delta_k L = \delta_k L - \frac{L_{|k} y^r}{L} \dot{\partial}_r L = \delta_k L - L_{|k} = 0.$$

Therefore the Finsler connection $*F\Gamma$ is h -metrical. Since $*F\Gamma$ is h -metrical and $h(h)$ -torsion $*T_{jk}^i$ is zero, the linear connection coefficients $*\Gamma_{jk}^i$ of $*F\Gamma$ are given in [1] by

$$(2.3) \quad \Gamma_{jk}^i = g^{ir} \Gamma_{jrk} = \frac{1}{2} g^{ir} [\delta_j g_{rk} + \delta_k g_{rj} - \delta_r g_{jk}].$$

Using (2.1) and $(\dot{\partial}_j g_{ik}) y^j = 0$ in (2.3), we have

$$(2.4) \quad *\Gamma_{jk}^i = \Gamma_{jk}^i.$$

Thus, the new connection $*F\Gamma = (*\Gamma_{jk}^i, *N_j^i, C_{jk}^i)$ reduces to $*F\Gamma = (\Gamma_{jk}^i, *N_j^i, C_{jk}^i)$.

The (v)hv-torsion tensor. The (v)hv-torsion tensor P_{jk}^i of a Finsler space with respect to connection $F\Gamma$ is defined by

$$P_{jk}^i = \dot{\partial}_k N_j^i - \Gamma_{jk}^i.$$

Therefore the (v)hv-torsion tensor ${}^*P_{jk}^i$ of a Finsler space with respect to the connection ${}^*F\Gamma$ is given by

$${}^*P_{jk}^i = \dot{\partial}_k^* N_j^i - {}^*\Gamma_{jk}^i.$$

Using (1.2) and (2.4), we get

$$(2.5) \quad {}^*P_{jk}^i = P_{jk}^i + l_{k|j} l^i + L_{|j} \frac{h_k^i}{L}.$$

Let us suppose ${}^*P_{jk}^i = P_{jk}^i$. Then equation (2.5) implies that $l_{k|j} l^i + L_{|j} \frac{h_k^i}{L} = 0$. Transvecting by y_i and using $h_k^i y_i = 0$, we get $L l_{k|j} = 0$, which gives $l_{k|j} = 0$.

Conversely, let $l_{k|j} = 0$. Also let us assume that deflection tensor D_j^i for the Finsler connection $F\Gamma$ is zero, i.e. $D_j^i = 0$. Then we get $y_{|j}^i = 0$. Again $l_{k|j} = 0$ and $y_{|j}^i = 0$, implies that $L_{|j} = 0$, and hence equation (2.5) yields ${}^*P_{jk}^i = P_{jk}^i$. Thus, we have:

Theorem 2.1. *Let the Finsler space F^n admits a Finsler connection $F\Gamma$ with zero deflection tensor and a Finsler connection ${}^*F\Gamma$. Then ${}^*P_{jk}^i = P_{jk}^i$ if and only if $l_{k|j} = 0$.*

Rephrasing, the theorem concludes that (v)hv-torsion tensors with respect to Finsler connections $F\Gamma$ and ${}^*F\Gamma$ are equal if and only if covariant differentiation of directional derivative of Fundamental metric function L vanishes.

The hv-curvature tensor. The hv-curvature tensor P_{hjk}^i of a Finsler space with respect to connection $F\Gamma$ is defined by

$$P_{hjk}^i = \dot{\partial}_k \Gamma_{hj}^i - C_{hk|j}^i + C_{hm}^i P_{jk}^m.$$

Therefore the hv-torsion tensor ${}^*P_{jk}^i$ of a Finsler space with respect to connection ${}^*F\Gamma$ is given by

$${}^*P_{hjk}^i = \dot{\partial}_k^* \Gamma_{hj}^i - C_{hk|j}^i + C_{hm}^i {}^*P_{jk}^m.$$

Using (2.4) and (2.5) we get

$$(2.6) \quad {}^*P_{hjk}^i = P_{hjk}^i + C_{hm}^i h_k^m \frac{L_{|j}}{L}.$$

Thus, we have:

Theorem 2.2. *Let the Finsler space F^n admits the Finsler connections $F\Gamma$ and ${}^*F\Gamma$. Then expression for ${}^*P_{hjk}^i$ is given by (2.6).*

Let us suppose that ${}^*P_{hjk}^i = P_{hjk}^i$. Then equation (2.6) implies that

$$C_{hm}^i h_k^m \frac{L_{|j}}{L} = 0,$$

which gives $C_{hm}^i = 0$, i.e. the space is Riemannian. Thus, we have:

Theorem 2.3. *If hv-curvature tensor with respect to Finsler connections $F\Gamma$ and ${}^*F\Gamma$ coincides, then the space will be Riemannian.*

The v(h)-torsion tensor. The v(h)-torsion tensor R_{jk}^i of a Finsler space with respect to connection $F\Gamma$ is defined by

$$R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i.$$

Therefore the v(h)-torsion tensor ${}^*R_{jk}^i$ of a Finsler space with respect to connection ${}^*F\Gamma$ is given by

$${}^*R_{jk}^i = {}^*\delta_k {}^*N_j^i - {}^*\delta_j {}^*N_k^i.$$

Using (1.2) and (2.1) and simplification gives us

$$(2.7) \quad {}^*R_{jk}^i = R_{jk}^i + [L_{|j|k} l^i + L_{|j|k} l_k^i - L_{|k} l^r l_{r|j} l^i - j/k],$$

where $-j/k$ denotes interchange of j and k and subtract the terms within the bracket.

Theorem 2.4. *Let the Finsler space F^n admits the Finsler connections $F\Gamma$ and ${}^*F\Gamma$. Then expression for ${}^*R_{jk}^i$ is given by (2.7).*

Equation (2.7) can be re-written as

$$(2.8) \quad {}^*R_{jk}^i = R_{jk}^i + [(L_{|j|k} - L_{|k|j}) l^i + L_{|j} \frac{y_{|k}^i}{L} - L_{|k} \frac{y_{|j}^i}{L} + L_{|k} \frac{y_{|j}^r l^r l^i}{L} - L_{|j} \frac{y_{|k}^r l^r l^i}{L}].$$

Let us assume that deflection tensor D_j^i for the Finsler connection $F\Gamma$ is zero i.e. $D_j^i = 0$. Then we get $y_{|j}^i = 0$ and then equation (2.8) becomes

$$(2.9) \quad {}^*R_{jk}^i = R_{jk}^i + (L_{|j|k} - L_{|k|j}) l^i.$$

Thus, we have:

Corollary 1. *Let the Finsler space F^n admits the Finsler connection $F\Gamma$ with zero deflection tensor and a Finsler connection ${}^*F\Gamma$. Then v(h)-torsion tensor ${}^*R_{jk}^i$ for the connection ${}^*F\Gamma$ is given by (2.9).*

The h-curvature tensor. The h-curvature tensor R_{hjk}^i of a Finsler space with respect to connection $F\Gamma$ is defined by

$$R_{hjk}^i = C_{hm}^i R_{jk}^m + \delta_k \Gamma_{hj}^i + \Gamma_{hj}^m \Gamma_{mk}^i - \delta_j \Gamma_{hk}^i - \Gamma_{hk}^m \Gamma_{mj}^i.$$

Therefore the h-curvature tensor ${}^*R_{jk}^i$ of a Finsler space with respect to connection ${}^*F\Gamma$ is given by

$${}^*R_{hjk}^i = C_{hm}^i {}^*R_{jk}^m + {}^*\delta_k \Gamma_{hj}^i + \Gamma_{hj}^m \Gamma_{mk}^i - {}^*\delta_j \Gamma_{hk}^i - \Gamma_{hk}^m \Gamma_{mj}^i.$$

Using (2.1) and (2.7) and simplifying, we obtain

$$(2.10) \quad {}^*R_{hjk}^i = R_{hjk}^i - (L_{|k}l^r\dot{\partial}_r\Gamma_{hj}^i - L_{|j}l^r\dot{\partial}_r\Gamma_{hk}^i) + C_{hm}^i[L_{|j}l_{|k}^m - L_{|k}l_{|j}^m].$$

Thus, we have:

Theorem 2.5. *Let the Finsler space F^n admits Finsler connections $F\Gamma$ and ${}^*F\Gamma$. Then expression for ${}^*R_{hjk}^i$ is given by (2.10).*

Equation (2.10) can be re-written as

$$(2.11) \quad {}^*R_{hjk}^i = R_{hjk}^i - L_{|k}l^r\dot{\partial}_r\Gamma_{hj}^i + L_{|j}l^r\dot{\partial}_r\Gamma_{hk}^i + C_{hm}^i[L_{|j}\frac{y_{|k}^m}{L} - L_{|k}\frac{y_{|j}^m}{L}].$$

Also let us assume that deflection tensor D_j^i for the Finsler connection $F\Gamma$ is zero, i.e. $D_j^i = 0$. Then we get $y_{|j}^i = 0$ and the equation (2.11) yields

$$(2.12) \quad {}^*R_{hjk}^i = R_{hjk}^i - (L_{|k}l^r\dot{\partial}_r\Gamma_{hj}^i) + (L_{|j}l^r\dot{\partial}_r\Gamma_{hk}^i).$$

Thus, we have:

Corollary 2. *Let the Finsler space F^n admits the Finsler connections $F\Gamma$ with deflection zero and ${}^*F\Gamma$. Then expression for ${}^*R_{hjk}^i$ is given by (2.12).*

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