

NON-DEGENERATE HYPERSURFACES OF A SEMI-RIEMANNIAN MANIFOLD WITH A QUARTER-SYMMETRIC NON-METRIC CONNECTION

AJIT BARMAN

ABSTRACT. The object of the present paper is to study a non-degenerate hypersurface of a semi-Riemannian manifold with a quarter-symmetric non-metric connection.

1. INTRODUCTION

Let $\tilde{M}$ be an $(n + 1)$ -dimensional differentiable manifold of class C^∞ and M an n -dimensional differentiable manifold immersed in $\tilde{M}$ by a differentiable immersion

$$i: M \rightarrow \tilde{M}.$$

$i(M)$ identical to M , is said to be a hypersurface of $\tilde{M}$. The differential di of the immersion i will be denoted by B so that a vector field X in M corresponds to a vector field BX in $\tilde{M}$. We suppose that the manifold $\tilde{M}$ is a semi-Riemannian manifold with the semi-Riemannian metric $\tilde{g}$ of index ν , $0 \leq \nu \leq n + 1$. Thus the index of $\tilde{M}$ is the ν , which will be denoted by $ind\tilde{M} = \nu$. If the induced metric tensor $g = \tilde{g}|_M$ defined by

$$g(X, Y) = \tilde{g}(BX, BY), \quad \text{for all } X, Y \text{ in } \chi(M)$$

is non-degenerate, then the hypersurface M is called a non-degenerate hypersurface. Also M is a semi-Riemannian manifold with the induced semi-Riemannian metric g [6]. If the semi-Riemannian manifolds $\tilde{M}$ and M are both orientable and we can choose a unit vector field N defined along M such that

$$\tilde{g}(BX, N) = 0, \quad \tilde{g}(N, N) = \epsilon = \begin{cases} +1, & \text{for spacelike } N, \\ -1, & \text{for timelike } N, \end{cases}$$

2010 *Mathematics Subject Classification.* 53C15, 53C25.

Key words and phrases. quarter-symmetric non-metric connection, Levi-Civita connection, Gauss curvature, Codazzi-Mainardi equation.

for all X in $\chi(M)$, where N is called the unit normal vector field to M and $\text{ind } M = \text{ind } \tilde{M}$ if $\epsilon = 1$, $\text{ind } M = \text{ind } \tilde{M} - 1$ if $\epsilon = -1$.

The hypersurface of a manifold have been studied by several authors such as Bucki [1], De and Kamilya [2], De and Mondal [3], O'Neill [6], Yücesan and Yasar [9], Yücesan and Ayyildiz [8], Yano and Kon [7] and many others.

In 1975, Golab [5] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A linear connection $\tilde{\nabla}$ on an $(n + 1)$ -dimensional Riemannian manifold $\tilde{M}$ is called a quarter-symmetric connection [5] if its torsion tensor $\tilde{T}$ satisfies

$$(1.1) \quad \tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\eta}(\tilde{Y})\tilde{\phi}\tilde{X} - \tilde{\eta}(\tilde{X})\tilde{\phi}\tilde{Y},$$

where $\tilde{\eta}$ is a 1-form and $\tilde{\xi}$ is a vector field defined by

$$(1.2) \quad \tilde{\eta}(\tilde{X}) = \tilde{g}(\tilde{X}, \tilde{\xi}),$$

for all vector fields $\tilde{X}, \tilde{Y} \in \chi(\tilde{M})$, $\chi(\tilde{M})$ is the set of all differentiable vector fields on $\tilde{M}$.

In particular, if $\tilde{\phi}\tilde{X} = \tilde{X}$, then the quarter-symmetric connection reduces to the semi-symmetric connection [4]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition

$$(1.3) \quad \tilde{\nabla}\tilde{g} \neq 0,$$

then $\tilde{\nabla}$ is said to be a quarter-symmetric non-metric connection.

2. QUARTER-SYMMETRIC NON-METRIC CONNECTION

Let $\tilde{M}$ denotes an $(n + 1)$ -dimensional semi-Riemannian manifold with semi-Riemannian metric $\tilde{g}$ of index ν , $0 \leq \nu \leq n + 1$. A linear connection $\tilde{\nabla}$ on $\tilde{M}$ is called a quarter-symmetric non-metric connection if

$$(2.1) \quad (\tilde{\nabla}_{\tilde{X}}\tilde{g})(\tilde{Y}, \tilde{Z}) = -\tilde{\eta}(\tilde{Y})\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Z}) - \tilde{\eta}(\tilde{Z})\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y}).$$

Throughout the paper, we will denote by $\tilde{M}$ the semi-Riemannian manifold admitting a quarter-symmetric non-metric connection given by

$$(2.2) \quad \tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\nabla}_{\tilde{X}}^*\tilde{Y} + \tilde{\eta}(\tilde{Y})\tilde{\phi}\tilde{X},$$

any vector fields $\tilde{X}$ and $\tilde{Y}$ of $\tilde{M}$. When M is a non-degenerate hypersurface, we have the following orthogonal direct sum :

$$(2.3) \quad \chi(\tilde{M}) = \chi(M) \oplus \chi(M)^\perp.$$

According to (2.3), every vector field $\tilde{X}$ on $\tilde{M}$ is decomposed as

$$(2.4) \quad \tilde{\xi} = B\xi + \mu N,$$

where μ is scalar function, a contravariant vector field ξ and a semi-Riemannian metric g on hypersurface M^n respectively.

We denote ∇^* the connection on the non-degenerate hypersurface M induced from the Levi-Civita connection $\tilde{\nabla}^*$ on $\tilde{M}$ with respect to the unit spacelike or timelike normal vector field N . We have the equality

$$(2.5) \quad \tilde{\nabla}_{BX}^* BY = B(\nabla_X^* Y) + h^*(X, Y)N,$$

for arbitrary vector fields X and Y of M , where h^* is the second fundamental form of the non-degenerate hypersurface M . Let us define the connection ∇ on M which is induced by the quarter-symmetric non-metric connection $\tilde{\nabla}$ on $\tilde{M}$ with respect to the unit spacelike or timelike normal vector field N . We obtain the equation

$$(2.6) \quad \tilde{\nabla}_{BX} BY = B(\nabla_X Y) + h(X, Y)N,$$

where h is the second fundamental form of the non-degenerate hypersurface M . We call (2.6) the equation of Gauss with respect to the induced connection ∇ .

According to (2.2), we have

$$(2.7) \quad \tilde{\nabla}_{BX} BY = \tilde{\nabla}_{BX}^* BY + \tilde{\eta}(BY)\tilde{\phi}(BX).$$

Using (2.5) and (2.6) in (2.7), we get

$$(2.8) \quad B(\nabla_X Y) + h(X, Y)N = B(\nabla_X^* Y) + h^*(X, Y)N + \tilde{\eta}(BY)\tilde{\phi}(BX),$$

which implies

$$(2.9) \quad \nabla_X Y = \nabla_X^* Y + \eta(Y)\phi X$$

and

$$h(X, Y) = h^*(X, Y),$$

where $\tilde{\eta}(BY) = B\eta(Y)$ and $\tilde{\phi}(BX) = B\phi X$.

From (2.9), we conclude that

$$(2.10) \quad (\nabla_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y),$$

and

$$(2.11) \quad T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

for any X, Y, Z in $\chi(M)$.

From (2.10) and (2.11), we can state the following theorem:

Theorem 2.1. *The connection induced on a non-degenerate hypersurface of a semi-Riemannian manifold with a quarter-symmetric non-metric connection with respect to the unit spacelike or timelike normal vector field is also a quarter-symmetric non-metric connection.*

The equation of Weingarten with respect to the Levi-Civita connection $\tilde{\nabla}^*$ is

$$(2.12) \quad \tilde{\nabla}_{BX}^* N = -B(A_N^* X),$$

for any vector field X in M and where A_N^* is a tensor field of type $(1, 1)$ of M which is defined by

$$(2.13) \quad h^*(X, Y) = \epsilon g(A_N^* X, Y),$$

[6].

Using (2.7), we have

$$(2.14) \quad \tilde{\nabla}_{BX} N = \tilde{\nabla}_{BX}^* N + \epsilon \mu \tilde{\phi}(BX),$$

because from (2.4), we obtain

$$\tilde{\eta}(N) = \tilde{g}(\tilde{\xi}, N) = \tilde{g}(B\xi + \mu N, N) = \mu \tilde{g}(N, N) = \epsilon \mu.$$

Combining (2.12) and (2.14), we get

$$(2.15) \quad \tilde{\nabla}_{BX} N = -B((A_N^* - \epsilon \mu \phi)X), \quad \epsilon \mp 1.$$

Applying the tensor field A_N of type $(1, 1)$ of M defined

$$(2.16) \quad A_N = A_N^* - \epsilon \mu \phi,$$

From (2.15), we have

$$(2.17) \quad \tilde{\nabla}_{BX} N = -B(A_N X),$$

Combining (2.13) and (2.16), we have

$$(2.18) \quad h(X, Y) = \epsilon g(A_N X, Y) + \mu g(\phi X, Y).$$

From (2.16), we can state the following corollary:

Corollary 2.1. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\tilde{M}$. Then*

(i) *if M has a spacelike normal vector field, the shape operator A_N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ is*

$$A_N = A_N^* - \mu \phi,$$

and

(ii) *if M has a timelike normal vector field, the shape operator A_N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ is*

$$A_N = A_N^* + \mu \phi.$$

We suppose that $e_1, e_2, \dots, e_\nu, e_{\nu+1}, \dots, e_n$ are principal vector fields corresponding to unit spacelike or timelike normal vector field N with respect to $\tilde{\nabla}^*$.

From (2.16), we have

$$A_N(e_i) = A_N^*(e_i) - \epsilon \mu \phi(e_i) = k_i^* e_i - \epsilon \mu p_i e_i = (k_i^* - \epsilon \mu p_i) e_i, \quad 1 \leq i \leq n,$$

where k_i^* , $1 \leq i \leq n$, are the principal curvatures corresponding to the unit spacelike or timelike normal vector field N with respect to the Levi-Civita connection $\tilde{\nabla}^*$. If we take

$$(2.19) \quad k_i = k_i^* - \epsilon \mu p_i \text{ and } \phi(e_i) = p_i e_i, \quad 1 \leq i \leq n,$$

then we obtain

$$(2.20) \quad A_N^*(e_i) = k_i^* e_i \text{ and } A_N(e_i) = k_i e_i, \quad 1 \leq i \leq n,$$

where k_i , $1 \leq i \leq n$, are the principal curvatures corresponding to the unit spacelike or timelike normal vector field N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$.

Therefore, we can give the following corollary:

Corollary 2.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\tilde{M}$. Then*

(i) *if M has a spacelike normal vector field, the principal curvatures corresponding to the unit spacelike normal vector field N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ are $k_i = k_i^* - \mu p_i$, $1 \leq i \leq n$,*

and

(ii) *if M has a timelike normal vector field, the principal curvatures corresponding to the unit timelike normal vector field N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ are $k_i = k_i^* + \mu p_i$, $1 \leq i \leq n$.*

3. EQUATIONS OF GAUSS CURVATURE AND CODAZZI-MAINARDI

We denote the curvature tensor of $\tilde{M}$ with respect to the Levi-Civita connection $\tilde{\nabla}^*$ by

$$\tilde{R}^*(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}^* \tilde{\nabla}_{\tilde{Y}}^* \tilde{Z} - \tilde{\nabla}_{\tilde{Y}}^* \tilde{\nabla}_{\tilde{X}}^* \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}^* \tilde{Z}$$

and that of M with respect to the Levi-Civita connection ∇^* by

$$R^*(X, Y)Z = \nabla_X^* \nabla_Y^* Z - \nabla_Y^* \nabla_X^* Z - \nabla_{[X, Y]}^* Z.$$

Then the equation of Gauss curvature is given by

$$\begin{aligned} R^*(X, Y, Z, U) &= \tilde{R}^*(BX, BY, BZ, BU) \\ &\quad + \epsilon\{h^*(X, U)h^*(Y, Z) - h^*(Y, U)h^*(X, Z)\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{R}^*(BX, BY, BZ, BU) &= \tilde{g}(\tilde{R}^*(BX, BY)BZ, BU), \\ R^*(X, Y, Z, U) &= g(R^*(X, Y)Z, U) \end{aligned}$$

and the equation of Codazzi-Mainardi [6] is given by

$$\tilde{R}^*(BX, BY, BZ, N) = \epsilon\{(\nabla_X^* h^*)(Y, Z) - (\nabla_Y^* h^*)(X, Z)\}.$$

We find the equation of Gauss curvature and Codazzi-Mainardi with respect to the quarter-symmetric non-metric connection. The curvature tensor of the quarter-symmetric non-metric connection $\tilde{\nabla}$ of $\tilde{M}$ is

$$(3.1) \quad \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}.$$

Putting $\tilde{X} = BX$, $\tilde{Y} = BY$, $\tilde{Z} = BZ$ in (3.1) and using (2.6) and (2.17), we get

$$\tilde{R}(BX, BY)BZ = B[R(X, Y)Z + h(X, Z)A_N Y - h(Y, Z)A_N X]$$

$$(3.2) \quad +[(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h\{\eta(Y)\phi X - \eta(X)\phi Y, Z\}]N,$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$$

is the curvature tensor of the quarter-symmetric non-metric connection ∇ .

Combining (2.18) and (3.2), we obtain

$$(3.3) \quad \begin{aligned} \tilde{R}(BX, BY, BZ, BU) &= R(X, Y, Z, U) + \epsilon[h(X, Z)h(Y, U) - h(Y, Z)h(X, U) \\ &\quad + \mu h(Y, Z)g(\phi X, U) - \mu h(X, Z)g(\phi Y, U)] \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} \tilde{R}(BX, BY, BZ, N) &= \epsilon[(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ &\quad + h\{\eta(Y)\phi X - \eta(X)\phi Y, Z\}], \end{aligned}$$

where $\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = \tilde{g}(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{U})$ and $R(X, Y, Z, U) = g(R(X, Y)Z, U)$.

The above equations (3.3) and (3.4) are Gauss curvature and Codazzi-Mainardi with respect to the quarter-symmetric non-metric connection respectively.

4. THE RICCI AND SCALAR CURVATURES

Suppose that $\{Be_1, \dots, Be_\nu, Be_{\nu+1}, \dots, Be_n, N\}$ is an orthonormal basis of $\chi(\tilde{M})$. Then the Ricci tensor of $\tilde{M}$ with respect to the quarter-symmetric non-metric connection is

$$(4.1) \quad \begin{aligned} \tilde{\text{Ric}}(BY, BZ) &= \sum_{i=1}^n \epsilon_i \tilde{g}(\tilde{R}(Be_i, BY)BZ, Be_i) \\ &\quad + \epsilon \tilde{g}(\tilde{R}(N, BY)BZ, N), \end{aligned}$$

for all X, Y in $\chi(M)$.

Using (2.18) in (3.3), we get

$$(4.2) \quad \begin{aligned} \tilde{R}(BX, BY, BZ, BU) &= R(X, Y, Z, U) + \epsilon g(A_N X, Z)g(A_N Y, U) \\ &\quad + \mu g(\phi X, Z)g(A_N Y, U) \\ &\quad - \epsilon g(A_N Y, Z)g(A_N X, U) - \mu g(A_N X, U)g(\phi Y, U). \end{aligned}$$

Putting $X = e_i$ and $U = e_i$ in (4.2) and using (2.19) and (2.20), we have

$$(4.3) \quad \begin{aligned} \sum_{i=1}^n \epsilon_i \tilde{R}(Be_i, BY, BZ, Be_i) &= \sum_{i=1}^n \epsilon_i \tilde{g}(\tilde{R}(Be_i, BX)BY, Be_i) = \text{Ric}(Y, Z) \\ &\quad + \sum_{i=1}^n [\epsilon k_i + \mu p_i - \epsilon f]g(A_N Y, Z) - \mu f g(\phi Y, Z), \end{aligned}$$

where

$$f = \sum_{i=1}^n \epsilon_i g(A_N e_i, e_i).$$

Using (4.3) in (4.1), we obtain

$$(4.4) \quad \begin{aligned} \tilde{\text{Ric}}(BY, BZ) = \text{Ric}(Y, Z) + \sum_{i=1}^n [\epsilon k_i + \mu p_i - \epsilon f] g(A_N Y, Z) - \mu f g(\phi Y, Z) \\ + \epsilon \tilde{g}(\tilde{R}(N, BY) BZ, N), \end{aligned}$$

where $\tilde{\text{Ric}}$ and Ric are the Ricci tensor with respect to $\tilde{\nabla}$ and ∇ respectively.

The scalar curvature of $\tilde{M}$ with respect to the quarter-symmetric non-metric connection is

$$(4.5) \quad \tilde{r} = \sum_{i=1}^n \epsilon_i \tilde{\text{Ric}}(e_i, e_i) + \epsilon \tilde{\text{Ric}}(N, N).$$

Putting $Y = e_i$ and $Z = e_i$ in (4.4), we get

$$(4.6) \quad \tilde{r} = r + \sum_{i=1}^n [\epsilon k_i + \mu p_i - \epsilon f] f - \mu f \alpha + 2\epsilon \tilde{\text{Ric}}(N, N),$$

where $\alpha = g(\phi e_i, e_i)$ and $\tilde{r}$ and r are the scalar curvature with respect to $\tilde{\nabla}$ and ∇ respectively.

From the above discussion, we can state the following theorem:

Theorem 4.1. *Let M be non-degenerate hypersurface of a semi-Riemannian manifold $\tilde{M}$ with respect to the quarter-symmetric non-metric connection, then*

- (i) *the Ricci tensor is given by (4.4)*
- and
- (ii) *the scalar curvature is given by (4.6).*

REFERENCES

- [1] A. Bucki. Hypersurfaces of almost r -paracontact Riemannian manifolds. *Tensor (N.S.)*, 48(3):245–251, 1989.
- [2] U. C. De and D. Kamilya. Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection. *J. Indian Inst. Sci.*, 75(6):707–710, 1995.
- [3] U. C. De and A. K. Mondal. Hypersurfaces of Kenmotsu manifolds endowed with a quarter-symmetric non-metric connection. *Kuwait J. Sci. Engrg.*, 39(1A):43–56, 2012.
- [4] A. Friedmann and J. A. Schouten. Über die Geometrie der halbsymmetrischen Übertragungen. *Math. Z.*, 21(1):211–223, 1924.
- [5] S. Gołąb. On semi-symmetric and quarter-symmetric linear connections. *Tensor (N.S.)*, 29(3):249–254, 1975.
- [6] B. O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity.
- [7] K. Yano and M. Kon. *Structures on manifolds*, volume 3 of *Series in Pure Mathematics*. World Scientific Publishing Co., Singapore, 1984.
- [8] A. Yücesan and N. Ayyildiz. Non-degenerate hypersurfaces of a semi-Riemannian manifold with a semi-symmetric metric connection. *Arch. Math. (Brno)*, 44(1):77–88, 2008.

- [9] A. Yücesan and E. Yaşar. Non-degenerate hypersurfaces of a semi-Riemannian manifold with a semi-symmetric non-metric connection. *Math. Rep. (Bucur.)*, 14(64)(2):209–219, 2012.

Received June 28, 2014.

DEPARTMENT OF MATHEMATICS,
KABI-NAZRUL MAHAVIDYALAYA,
P.O.:-SONAMURA-799181,
P.S.-SONAMURA,
DIST- SEPAHIJALA, TRIPURA, INDIA
E-mail address: `ajitbarmanaw@yahoo.in`