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MODULES OVER GROUP RINGS OF LOCALLY FINITE GROUPS WITH FINITENESS RESTRICTIONS

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Abstract. We study an $\mathbf{R}G$-module A , where $\mathbf{R}$ is a ring, $A/C_A(G)$ is infinite, $C_G(A) = 1$, G is a group. Let $\mathfrak{L}_{\mathrm{nf}}(G)$ be the system of all subgroups $H \leq G$ such that the quotient modules $A/C_A(H)$ are infinite. We investigate an $\mathbf{R}G$-module A such that $\mathfrak{L}_{\mathrm{nf}}(G)$ satisfies either the weak minimal condition or the weak maximal condition as an ordered set. It is proved that if G is a locally finite group then either G is a Chernikov group or G is a finite-finitary group of automorphisms of A.

1. Introduction

Important finiteness conditions in group theory are the weak minimal condition on subgroups and the weak maximal condition on subgroups. Let G be a group, $\mathcal{M}$ be a set of subgroups of G. G is said to satisfy the weak minimal condition on $\mathcal{M}$-subgroups if for a descending series of subgroups $G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n \geq G_{n+1} \geq \cdots$, $G_n \in \mathcal{M}$, $n \in \mathbb{N}$, there exists the number $m \in \mathbb{N}$ such that an index $|G_n : G_{n+1}|$ is finite for any $n \geq m$ [11]. Similarly G is said to satisfy the weak maximal condition on $\mathcal{M}$-subgroups if for an ascending series of subgroups $G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n \leq G_{n+1} \leq \cdots$, $G_n \in \mathcal{M}$, $n \in \mathbb{N}$, there exists the number $m \in \mathbb{N}$ such that an index $|G_n : G_{n+1}|$ is finite for any $n \geq m$ [1].

These finiteness conditions were applied to investigate infinite dimensional linear periodic groups [9]. Also similar finiteness conditions were considered in [2].

Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, G be a group. G is a finite-finitary group of automorphisms of A if $C_G(A) = 1$ and $A/C_A(g)$ is finite for any $g \in G$ [10]. Finite-finitary groups of automorphisms of A with additional restrictions were studied in [10].

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Let $\mathfrak{L}_{\mathrm{nf}}(G)$ be the system of all subgroups H of G such that $A/C_A(H)$ is infinite. Previously, we studied an $\mathbf{R}G$-module A with some restrictions on subgroups of $\mathfrak{L}_{\mathrm{nf}}(G)$ [4, 3, 5].

In this paper we continue these investigations. We say that G satisfies the condition $W_{\text{min-nf}}$ if G satisfies the weak minimal condition on $\mathcal{M}$-subgroups where $\mathcal{M} = \mathfrak{L}_{\mathrm{nf}}(G)$ and G satisfies the condition $W_{\text{max-nf}}$ if G satisfies the weak maximal condition on $\mathcal{M}$-subgroups where $\mathcal{M} = \mathfrak{L}_{\mathrm{nf}}(G)$.

Next, we consider an $\mathbf{R}G$-module A with $C_G(A) = 1$. We investigate an $\mathbf{R}G$-module A such that G satisfies either $W_{\text{min-nf}}$ or $W_{\text{max-nf}}$. Main results of the paper are Theorem 1 and Theorem 2.

2. Preliminary results

Lemma 1. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring. Then the following conditions hold:*

(1) if $L \leq H \leq G$ and $A/C_A(H)$ is finite then $A/C_A(L)$ is finite also;

(2) if $L, H \leq G$, $A/C_A(L)$ and $A/C_A(H)$ are finite then $A/C_A(\langle L, H\rangle)$ is finite also.

Corollary 1. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, $FFD(G)$ be the set of all elements $x \in G$ such that $A/C_A(x)$ is finite. Then $FFD(G)$ is a normal subgroup of G.*

Proof. By Lemma 1 (2) $FFD(G)$ is a subgroup of G. Since $C_A(x^g) = C_A(x)g$ for each $x, g \in G$ then $FFD(G)$ is a normal subgroup of G. □

Lemma 2. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, H be a subgroup of G. Suppose that H contains a normal subgroup K such that $A/C_A(K)$ is infinite. Then the following conditions hold:*

(1) if G satisfies $W_{\text{min-nf}}$ then H/K satisfies the weak condition of minimality on subgroups;

(2) if G satisfies $W_{\text{max-nf}}$ then H/K satisfies the weak condition of maximality on subgroups.

Lemma 3. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, L, K and H be subgroups of G with the the following properties:*

(i) K is a normal subgroup of L;

(ii) K and L are H-invariant subgroups;

(iii) $L/K \cap HK/K = \langle 1\rangle$;

(iv) $L/K = Dr_{n\in\mathbb{N}}\, L_n/K$, $L_n/K \neq \langle 1\rangle$ is an H-invariant subgroup for any $n \in \mathbb{N}$.

Then the following conditions hold:

(1) if G satisfies $W_{\text{max-nf}}$ then $A/C_A(HL)$ is finite;

(2) if G satisfies $W_{\text{min-nf}}$ then $A/C_A(HK)$ is finite.

Proof. There are two infinite subsets Σ and Δ of $\mathbb{N}$ such that $\Sigma \cup \Delta = \mathbb{N}$, $\Sigma \cap \Delta = \emptyset$. Since Δ is infinite then there is an infinite strongly ascending series of subsets of Δ

$$\Delta(1) \subset \Delta(2) \subset \cdots \subset \Delta(k) \subset \cdots .$$

Also there is strongly descending series of subsets of Δ

$$\Delta^*(1) \supset \Delta^*(2) \supset \cdots \supset \Delta^*(k) \supset \cdots ,$$

such that the sets $\Delta(k+1) \setminus \Delta(k)$ and $\Delta^*(k) \setminus \Delta^*(k+1)$ are infinite for any $n \in \mathbb{N}$. Let

$$D_k/K = Dr_{t \in \Sigma \cup \Delta(k)} L_t/K$$

and

$$D_k^*/K = Dr_{t \in \Sigma \cup \Delta^*(k)} L_t/K.$$

At first we consider the strongly ascending series of subgroups

$$HD_1 < HD_2 < \cdots < HD_k < \cdots .$$

$|HD_{k+1} : HD_k|$ are infinite by construction. If G satisfies $W_{\text{max-nf}}$ then there is $m \in \mathbb{N}$ such that $A/C_A(HD_m)$ is finite. Since $\langle H, L_t | t \in \Sigma\rangle \le HD_m$ then $A/C_A(\langle H, L_t | t \in \Sigma\rangle)$ is finite by Lemma 1. Similarly we prove that $A/C_A(\langle H, L_t | t \in \Delta\rangle)$ is finite.

Since $\Sigma \cup \Delta = \mathbb{N}$ we obtain

$$\langle \langle H, L_t | t \in \Delta\rangle, \langle H, L_t | t \in \Sigma\rangle\rangle = \langle H, L_t | t \in \Sigma \cup \Delta\rangle = HL.$$

By Lemma 1 $A/C_A(HL)$ is finite.

Likewise we can construct the strongly descending series of subgroups

$$HD_1^* > HD_2^* > \cdots > HD_k^* > \cdots ,$$

such that $|HD_k^* : HD_{k+1}^*|$ are infinite. If G satisfies $W_{\text{min-nf}}$ then there is $m \in \mathbb{N}$ such that $A/C_A(HD_m^*)$ is finite. Since $HK \le HD_m^*$ then $A/C_A(HK)$ is finite by Lemma 1. □

Corollary 2. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, L, K and H be subgroups of G with the the following properties:*

(i) K is a normal subgroup of L;
(ii) K and L are H-invariant subgroups;
(iii) $L/K = Dr_{n \in \mathbb{N}}\, L_n/K$ where $L_n/K \neq \langle 1\rangle$ is an H-invariant subgroup for any $n \in \mathbb{N}$;
(iv) the set $\mathbb{N} \setminus \operatorname{Supp}(L/K \cap HK/K)$ is infinite.

If G satisfies either $W_{\text{min-nf}}$ or $W_{\text{max-nf}}$ then $A/C_A(HK)$ is finite. In particular $A/C_A(H)$ is finite.

Proof. Let $\Delta = \mathbb{N} \setminus \operatorname{Supp}(L/K \cap HK/K)$ and $T/K = Dr_{n \in \Delta}\, L_n/K$. Then $T/K \cap HK/K = \langle 1\rangle$. We apply Lemma 3. □

Corollary 3. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, L, K and H be subgroups of G with the the following properties:*

(i) K is a normal subgroup of L;
(ii) K and L are H-invariant subgroups;

(iii) $L/K = Dr_{n\in\mathbb{N}}\ L_n/K$, $L_n/K \neq \langle 1\rangle$ *is an* H*-invariant subgroup for any* $n \in \mathbb{N}$.

If G *satisfies either* $W_{min\text{-}nf}$ *or* $W_{max\text{-}nf}$ *then* $A/C_A(\langle h\rangle K)$ *is finite for any* $h \in H$. *In particular* $H \leq FFD(G)$.

Proof. Let $h \in H$. Since L_n/K is an H-invariant subgroup for any $n \in \mathbb{N}$ then L_n/K is an $\langle h\rangle$-invariant subgroup for any $n \in \mathbb{N}$. In particular the set $\mathrm{Supp}(\langle h\rangle K/K \cap L/K)$ is finite. Then $A/C_A(\langle h\rangle K)$ is finite by Corollary 2. □

3. Main results

Obviously that a Chernikov group satisfies both the weak minimal condition on subgroups and the weak maximal condition on subgroups. It follows that if A is an $\mathbf{R}G$-module and G is Chernikov then G satisfies both $W_{\text{min-nf}}$ and $W_{\text{max-nf}}$.

Lemma 4. *Let* A *be an* $\mathbf{R}G$*-module,* $\mathbf{R}$ *be an associative ring. Suppose that* G *satisfies either* $W_{min\text{-}nf}$ *or* $W_{max\text{-}nf}$. *Let* K *and* H *be subgroups of* G *such that* K *is a normal subgroup of* H *and* H/K *is an infinite elementary abelian* p*-group for some prime* p. *Suppose that* K *and* H *are* $\langle g\rangle$*-invariant for some* $g \in G$. *If* $g^k \in C_G(H/K)$ *for some* $k \in \mathbb{N}$ *then* $g \in FFD(G)$.

Proof. Let $M = H/K$. We take $1 \neq b_1 \in M$. Put $B_1 = \langle b_1\rangle^{\langle g\rangle}$. Since g induces the automorphism of finite order on M then B_1 is finite. $M = B_1 \times C_1$ is valid for some subgroup C_1.

Let

$$\{C_1^y | y \in \langle g\rangle\} = \{U_1, \dots, U_m\}.$$

It follows that the $\langle g\rangle$-invariant subgroup

$$D_1 = U_1 \cap \dots \cap U_m = Core_{\langle g\rangle}(C_1)$$

has finite index in M. Let $1 \neq b_2 \in D_1$ and $B_2 = \langle b_2\rangle^{\langle g\rangle}$. Then $\langle B_1, B_2\rangle = B_1 \times B_2$. As before we conclude that $M = (B_1 \times B_2) \times C_2$ for some subgroup C_2. Similarly we can construct the infinite set $\{B_n | n \in \mathbb{N}\}$ of non-trivial $\langle g\rangle$-invariant subgroups such that $\langle B_n | n \in \mathbb{N}\rangle = Dr_{n\in\mathbb{N}} B_n$. By Corollary 3 we have that $g \in FFD(G)$. □

Let $\pi(G)$ be the set of all prime divisors of orders of elements of G.

Corollary 4. *Let* A *be an* $\mathbf{R}G$*-module,* $\mathbf{R}$ *be an associative ring. Suppose that* G *satisfies either* $W_{min\text{-}nf}$ *or* $W_{max\text{-}nf}$. *Let* K *and* H *are subgroups of* G *such that* K *is a normal subgroup of* H, H/K *is a periodic almost locally solvable group. If* H/K *is not Chernikov then* $H \leq FFD(G)$.

Proof. Let L/K be a locally solvable normal subgroup of H/K of finite index. Since H/K is not Chernikov then L/K is not Chernikov too. Let g be an element of H. Then L/K contains an abelian $\langle g\rangle$-subgroup C/K which is not Chernikov [12]. If the set $\pi(C/K)$ is infinite then $g \in FFD(G)$ by Corollary

3. If $\pi(C/K)$ is finite then there is the prime p such that Sylow p-subgroup P/K of C/K is not Chernikov. It follows that the lower layer B/K of P/K is infinite. Therefore L/K contains a $\langle g\rangle$–invariant infinite elementary abelian subgroup B_1/K. Then $g \in FFD(G)$ by Lemma 4. □

Corollary 5. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring. Suppose that G satisfies either $W_{min\text{-}nf}$ or $W_{max\text{-}nf}$. Let K and H be subgroups of G such that K is a normal subgroup of H, H/K is a locally finite group. If H/K is not Chernikov then $H \leq FFD(G)$.*

Proof. Let g be an element of H and $C/K = C_{H/K}(gK)$. If C/K is not Chernikov then by Theorem [8] C/K contains an abelian subgroup D/K which is a direct product of infinite set of non-trivial cyclic subgroups. By Corollary 3 we have that $g \in FFD(G)$. Suppose that C/K is not Chernikov. Then H/K is an almost locally solvable group [6] and $g \in FFD(G)$ by Corollary 4. We have that $H \leq FFD(G)$. □

It follows that Theorem 1 is valid.

Theorem 1. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, G be a locally finite group. If G satisfies either $W_{min\text{-}nf}$ or $W_{max\text{-}nf}$ then either G is a Chernikov group or G is a finite-finitary group of automorphisms of A.*

Lemma 5. *Let A be a $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, G be a locally solvable group. Suppose that $A/C_A(G)$ is finite. Then G is almost abelian.*

Proof. Let $C = C_A(G)$. Then A has the series of $\mathbf{R}G$-submodules $\langle 0\rangle \leq C \leq A$, where A/C is a finite $\mathbf{R}$-module. Since $G \leq C_G(C)$ then $G/C_G(C)$ is trivial. As A/C is a finite $\mathbf{R}$-module then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. Each element of H acts trivially on every factor of the series $\langle 0\rangle \leq C \leq A/C$. By Kaluzhnin Theorem (p. 144 [7]) H is abelian. By Remak's Theorem

$$G/H \leq G/C_G(C) \times G/C_G(A/C).$$

It follows that G/H is finite. Then G is an almost abelian group. □

Let $G_{\mathfrak{S}}$ be the intersection of all normal subgroups K of G such that G/K is solvable. If G is a solvable group then we denote the step of solvability of G by s(G).

Theorem 2. *Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, G be a locally solvable periodic group. If G satisfies either $W_{min\text{-}nf}$ or $W_{max\text{-}nf}$ then $G/G_{\mathfrak{S}}$ is solvable.*

Proof. Otherwise $H = G/G_{\mathfrak{S}}$ is unsolvable. Let F_1 be a finite subgroup of H. Since H is approximated by solvable subgroups then there is a normal subgroup K_1 of H such that $F_1 \cap K_1 = \langle 1\rangle$ and H/K_1 is solvable. It follows that K_1 is unsolvable. Therefore the steps of solvability of finite subgroups of K_1 not limited by the number. Then K_1 contains a finite subgroup D_1 such that $s(F_1) < s(D_1)$. Since F_1 and D_1 are finite then they are solvable. Let $F_2 = D_1^{F_1}$. Then F_2 is a finite F_1-invariant subgroup such that $s(F_1) < s(F_2)$. Since F_1F_2 is finite there is a normal subgroup K_2 of H such that $F_1F_2 \cap K_2 = \langle 1 \rangle$ and H/K_2 is solvable. Since K_2 is unsolvable then we can choose a finite F_1F_2-invariant subgroup F_3 of K_2 such that $s(F_2) < s(F_3)$. Continuing our reasoning, we construct the strongly ascending series of finite subgroups $F_1 < F_1F_2 < \cdots < F_1F_2\cdots F_n < \cdots$ with the the following properties:

(i) F_n is an F_j-invariant subgroup for $j < n$;
(ii) $s(F_j) < s(F_n)$ for $j < n$;
(iii) $F_1F_2\cdots F_n \cap \langle F_j | j > n\rangle = \langle 1 \rangle$ for any $n \in \mathbb{N}$.

It follows that $\langle F_j | j \in \Delta\rangle$ is decomposed in the direct product of $F_j, j \in \Delta$, for an infinite subset Δ of $\mathbb{N}$. Therefore $\langle F_j | j \in \Delta\rangle$ is unsolvable.

At first we suppose that G satisfies $W_{\text{min-nf}}$. There is an infinite strictly descending series of subsets

$$\mathbb{N} \supset \Delta(1) \supset \Delta(1) \supset \cdots \supset \Delta(k) \supset \cdots$$

such that $\Delta(k) \setminus \Delta(k+1)$ is infinite for any $k \in \mathbb{N}$. Let $L_k = \langle F_j | j \in \Delta(k)\rangle$ for any $k \in \mathbb{N}$. We obtain the strongly descending series of subgroups $L_1 > L_2 > \cdots > L_k > \cdots$ of H. Let M_k is the preimage of L_k in G. Then $M_1 > M_2 > \cdots > M_k > \cdots$ is the strongly descending series of subgroups of G such that $|M_k : M_{k+1}|$ are infinite. Hence there is $t \in \mathbb{N}$ such that $A/C_A(M_t)$ is finite. M_t is solvable by Lemma 5. It follows that $L_t = M_t/G_{\mathfrak{S}}$ is solvable. Previously, we proved that $L_t = \langle F_j | j \in \Delta(t)\rangle$ is unsolvable. Contradiction.

If G satisfies $W_{\text{max-nf}}$ we construct an infinite strictly ascending series of subsets of $\mathbb{N}$ and conduct similar reasoning. $\square$

When writing the paper the author used the methods of [9].

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