

ON SOME LOCAL PROPERTIES OF THE CONJUGATE FUNCTION AND THE MODULUS OF CONTINUITY OF k -TH ORDER

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ABSTRACT. In the present paper we study a local smoothness of the conjugate functions of several variables in the space $\mathbb{C}(T^n)$. The direct estimates are obtained and exactness of these estimates are established by proper examples.

1. INTRODUCTION

Let $\mathbb{R}^n$ ($n = 1, 2, \dots$; $\mathbb{R}^1 \equiv \mathbb{R}$) be the n -dimensional Euclidean space of points $\bar{x} = (x_1, \dots, x_n)$ with real coordinates. Let B be an arbitrary non-empty subset of the set $M = \{1, \dots, n\}$. Denote by $|B|$ the cardinality of B . Let x_B be such a point in $\mathbb{R}^n$ whose coordinates with indices in $M \setminus B$ are zero.

As usual $\mathbb{C}(T^n)$ ($\mathbb{C}(T^1) \equiv \mathbb{C}(T)$), where $T = [-\pi, \pi]$, denotes the space of all continuous functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ that are 2π -periodic in each variable, endowed with the norm

$$\|f\| = \max_{\bar{x} \in T^n} |f(\bar{x})|.$$

If $f \in L(T^n)$, then following Zhizhiashvili [10], we call the expression

$$\tilde{f}_B(\bar{x}) = \left(-\frac{1}{2\pi}\right)^{|B|} \int_{T^{|B|}} f(\bar{x} + s_B) \prod_{i \in B} \cot \frac{s_i}{2} ds_B$$

the conjugate function of n variables with respect to those variables whose indices form the set B (with $\tilde{f}_B \equiv \tilde{f}$ for $n = 1$).

Suppose that $f \in \mathbb{C}(T^n)$, $1 \leq i \leq n$, and $h \in T$. Then for each $\bar{x} \in T^n$ let us consider the difference of k -th order

$$\Delta_i^k(h) f(\bar{x}) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(x_1, \dots, x_{i-1}, x_i + jh, x_{i+1}, \dots, x_n)$$

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and define the partial modulus of continuity of k -th order of the function f with respect to the variable x_i by the equality

$$\omega_{k,i}(f; \delta) = \sup_{|h| \leq \delta} \|\Delta_i^k(h) f\|.$$

$(\Delta_i^k(h) f(x) \equiv \Delta^k(h) f(x)$ and $\omega_{k,i}(f; \delta) \equiv \omega_k(f; \delta)$ for $n = 1$).

Definition 1. We say that a function φ is almost decreasing in $[a, b]$ if there exists a positive constant A such that $\varphi(t_1) \geq A \varphi(t_2)$ for $a \leq t_1 \leq t_2 \leq b$.

Definition 2. A function $\omega_k: [0, \pi] \rightarrow \mathbb{R}$ which satisfies the following four conditions:

- (1) $\omega_k(0) = 0$,
- (2) ω_k is nondecreasing,
- (3) ω_k is continuous,
- (4) $\frac{\omega_k(t)}{t^k}$ is almost decreasing in $[0, \pi]$,

we call the modulus of continuity of k -th order.

Definition 3. We say that the modulus of continuity of k -th order ω_k satisfies Zygmund's condition if

$$\int_0^\delta \frac{\omega_k(t)}{t} dt + \delta^k \int_\delta^\pi \frac{\omega_k(t)}{t^{k+1}} dt = O(\omega_k(\delta)), \quad \delta \rightarrow 0+.$$

Let ω_k be a modulus of continuity of k -th order. Then we denote by $H_i(\omega_k; \mathbb{C}(T^n))$ ($i = 1, \dots, n$) the set of all functions $f \in \mathbb{C}(T^n)$ such that

$$\omega_{k,i}(f; \delta) = O(\omega_k(\delta)), \quad \delta \rightarrow 0+, \quad i = 1, \dots, n.$$

We set

$$H(\omega_k; \mathbb{C}(T^n)) = \bigcap_{i=1}^n H_i(\omega_k; \mathbb{C}(T^n)).$$

By I we denote the following subset of the set $\mathbb{R}^n$:

$$\{\bar{x} : \bar{x} = \underbrace{(x, \dots, x)}_n; x \in T\}.$$

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f , they essentially measure the structure or smoothness of the function via the k -th difference $\Delta_i^k(h) f(\bar{x})$. In fact, for the functions f belonging to the Lebesgue space L^p ($1 \leq p < +\infty$) or the space of continuous functions $\mathbb{C}$, the classical k -th modulus of continuity has turned out to be a rather good measure for determining the rate of convergence of best approximation. On this direction one could see books by V. K. Dzyadyk, I. A. Shevchuk [4] and by R. Trigub, E. Belinsky [9].

In the theory of functions of real variables there is a well-known theorem of Privalov on the invariance of the functional class $\text{Lip}(\alpha, C(T))$ ($0 < \alpha < 1$) under the conjugate function $\tilde{f}$. If $\alpha = 1$ the invariance of the functional class

fails. Later Zygmund [10] established that the analogous theorem is valid in the case $\alpha = 1$ for the modulus of continuity of the second order. Afterwards Bari and Stechkin obtained results connected with behaviour of the moduli of continuity of k -th order of the function f and its conjugate function. They obtained the necessary and sufficient condition on the modulus of continuity of k -th order ω_k for the invariance of $H(\omega_k; \mathbb{C}(T))$ class under the conjugate function $\tilde{f}$. As to the functions of many variables, the first result in this direction belongs to Cesari and Zhak. They showed that the class $\text{Lip}(\alpha, C(T^2))$ ($0 < \alpha < 1$) is not invariant under the conjugate operators of two variables. Later, there were obtained the sharp estimates for partial moduli of continuity of different orders in the space of continuous functions [2, 3, 7]. The cases when moduli of continuity of different orders satisfy Zygmund's condition were considered in works [1, 5, 6]. In the present work, we study the behaviour of the smoothness of the conjugate functions $\tilde{f}_B$ on the set I . If we restrict the function $\tilde{f}_B$ on the set I , we can consider it as a function of one variable. The following question arises: what we can say about the smoothness of this 'new function' if the function f belongs to $H(\omega_k; \mathbb{C}(T^n))$ and the modulus of continuity ω_k satisfies Zygmund's condition.

We now state the facts on which the proof of the main results is based.

Lemma 1 (see [3, p. 283]). *Let ω_k be a modulus of continuity of k -th order, I_l be a system of pairwise disjoint intervals, $I_l \subset T$ for each l ($l = 1, 2, \dots$). Let $(f_l)_{l \geq 1}$ be a sequence of functions such that for each l , $f_l \in \mathbb{C}(T)$ and $f_l(x) = 0$ when $x \in T \setminus I_l$. If*

$$\omega_k(f_l; \delta) \leq \omega_k(\delta), \quad 0 \leq \delta \leq \pi, \quad l = 1, 2, \dots$$

and the function f is defined by the equality $f(x) = \sum_{l=1}^{\infty} f_l(x)$, then

$$\omega_k(f; \delta) \leq (k+1) \omega_k(\delta), \quad 0 \leq \delta \leq \pi.$$

Note that the case $k = 1$ is considered in [8, Lemma 1].

Remark 1. [3, p. 285] By the definition of the partial modulus of continuity of the multivariable function $f \in \mathbb{C}(T^n)$, it is easy to obtain the multivariable versions of Lemma 1 for partial moduli of continuity.

2. MAIN RESULTS

We can state and prove the following Theorem.

Theorem 1. a) *Let $f \in H(\omega_k, \mathbb{C}(T^n))$ and modulus of continuity of k -th order satisfies Zygmund's condition. Then*

$$(1) \quad \sup_{\bar{h} \in I, |h| \leq \delta} \sup_{\bar{x} \in I} |\Delta_j^k(h) \tilde{f}_B(\bar{x})| = O(\omega_k(\delta) |\ln \delta|^{|B|-1}), \quad j \in B, \quad \delta \rightarrow 0+,$$

$$(2) \quad \sup_{\bar{h} \in I, |h| \leq \delta} \sup_{\bar{x} \in I} |\Delta_j^k(h) \tilde{f}_B(\bar{x})| = O(\omega_k(\delta) |\ln \delta|^{|B|}), \quad j \in M \setminus B, \quad \delta \rightarrow 0+.$$

b) For each $B \subseteq M$ there exist functions F and G such that $F, G \in H(\omega_k; \mathbb{C}(T^n))$ and

$$(3) \quad \sup_{\bar{h} \in I, |h| \leq \delta} \sup_{\bar{x} \in I} |\Delta_j^k(h) \tilde{F}_B(\bar{x})| \geq C\omega_k(\delta) |\ln \delta|^{B-1}, \quad j \in B, \quad 0 \leq \delta \leq \delta_0,$$

$$(4) \quad \sup_{\bar{h} \in I, |h| \leq \delta} \sup_{\bar{x} \in I} |\Delta_j^k(h) \tilde{G}_B(\bar{x})| \geq C\omega_k(\delta) |\ln \delta|^{B|}, \quad j \in M \setminus B, \quad 0 \leq \delta \leq \delta_0,$$

where C and δ_0 are positive constants.

Proof. a) Part a) is the particular case of the first part of the theorem given in [3].

b) Without loss of generality, we shall carry out the proof of part (b) for the case $B = \{1, \dots, m\}$ ($1 \leq m \leq n$).

Let first $B = \{1, \dots, m\}$ ($1 \leq m < n$). Let us consider a strictly decreasing sequence of positive numbers $(b_l)_{l \geq 1}$ such that

$$\sum_{l=0}^{\infty} b_l \leq 1 \quad (b_0 = 0).$$

We set

$$\begin{aligned} \tau_p &= 2 \sum_{j=0}^{p-1} \omega_k^{-1}(b_j), \\ \tau_p^* &= \tau_p + \frac{2}{k} \omega_k^{-1}(b_p), \\ \tau_{p,q} &= \tau_p + q(\tau_p^* - \tau_p), \quad q = 2, \dots, k-1; p = 1, 2, \dots, \end{aligned}$$

where $\omega_k^{-1}(b_p)$, ($p = 1, 2, \dots$) is a certain element of the set $\{t : \omega_k(t) = b_p\}$.

Let $\tau_p \equiv \tau_{p,1}$ and $\tau_{p+1} \equiv \tau_{p,k}$. We define the functions $g_{p,q}$ and h_p ($p = 1, 2, \dots; q = 1, \dots, k-1$) in T as follows:

$$g_{p,q}(x) = \begin{cases} \frac{(x - \tau_{p,q})^k (\tau_{p,q+1} - x)^k}{(\tau_p^* - \tau_p)^{2k}}, & x \in [\tau_{p,q}; \tau_{p,q+1}], \\ 0, & \text{otherwise.} \end{cases}$$

$$h_p(x) = \begin{cases} 0, & x \in [-\pi; 2\tau_p - \tau_p^*], \\ \frac{(x + \tau_p^* - 2\tau_p)^k}{(\tau_p^* - \tau_p)^k}, & x \in (2\tau_p - \tau_p^*; \tau_p], \\ 1, & x \in (\tau_p; \pi - \tau_p^* + \tau_p] \\ \frac{(\pi - x)^k}{(\tau_p^* - \tau_p)^k}, & x \in (\pi - \tau_p^* + \tau_p; \pi]. \end{cases}$$

We define the function $G_{p,q}$ ($p = 1, 2, \dots; q = 1, \dots, k-1$) in T^n as follows:

$$G_{p,q}(x_1, \dots, x_n) = b_p \prod_{i=1}^m h_p(x_i) g_{p,q}(x_{m+1})$$

Consider the function G defined by the series

$$G(x_1, \dots, x_n) = \sum_{p=1}^{\infty} \sum_{q=1}^{k-1} G_{p,q}.$$

We extend this function G 2π -periodically in each variable to the whole space $\mathbb{R}^n$.

We claim that

$$G \in H(\omega_k; \mathbb{C}(T^n)).$$

It is known [4, p. 195] that if the function of one variable f has k th derivative on $(x, x + kh)$ then

$$\Delta_k(h) f(x) = h^k f^{(k)}(x + k\theta h), \quad 0 < \theta < 1.$$

In our situation using the definition of the function $G_{p,q}$ and this fact we can conclude that

$$|\Delta_i^k(G_{p,q}; h)| \leq D_1 |h^k| \frac{b_p}{(\tau_p^* - \tau_p)^k}, \quad D_1 = \text{const}, \quad i = 1, \dots, m+1.$$

Using this fact and the fact that $\frac{\omega_k(t)}{t^k}$ is almost decreasing we get

$$\omega_{k,i}(G_{p,q}; \delta) \leq D_2 \omega_k(\delta), \quad \delta \rightarrow 0+, \quad D_2 = \text{const}.$$

By Remark for Lemma 1 we conclude

$$\omega_{k,i}(G; \delta) = O(\omega_k(\delta)), \quad \delta \rightarrow 0+, i = 1, \dots, m+1.$$

If $i \in m+2, \dots, n$ then it is easy to conclude that

$$\omega_{k,i}(G; \delta) = O(\omega_k(\delta)), \quad \delta \rightarrow 0+.$$

Hence

$$G \in H(\omega_k; \mathbb{C}(T^n)).$$

Let $h = \tau_p^* - \tau_p$ and $x_i = \tau_p, i = 1, \dots, n$.

According to the definition of the function G we obtain

$$\begin{aligned} |\Delta_n^k(h) \tilde{G}_{\{1, \dots, m\}}(\tau_p, \dots, \tau_p)| &\geq D_3 b_p \int_{[\tau_p^* - \tau_p, 1]^m} \prod_{i=1}^m s_i^{-1} ds_i \\ &= D_3 \omega_k(\tau_p^* - \tau_p) |\ln(\tau_p^* - \tau_p)|^m, \quad D_3 = \text{const}. \end{aligned}$$

Therefore, the inequality (4) is proved.

To prove the inequality (3) we use the function F considered in [3, p. 289]

$$F(x) = F(x_1, x_2, \dots, x_n) =$$

$$\begin{cases}
\prod_{i=1}^m x_i^{k-1} (\pi - x_i)^{k-1} \int_{x_1(\pi-x_1)}^{2x_1(\pi-x_1)} \dots \int_{x_m(\pi-x_m)}^{2x_m(\pi-x_m)} \frac{\min_{1 \leq i \leq m} \omega_k(t_i)}{\prod_{i=1}^m t_i^k} \times & x_i \in [0, \pi], \ i = 1, \dots, m, \\
\times \prod_{i=1}^m \frac{(t_i - x_i(\pi - x_i))^k}{t_i^k} \frac{(2x_i(\pi - x_i) - t_i)^k}{t_i^k} dt, & x_j \in [-\pi, \pi], \\
0, & j = m+1, \dots, n, \\
& \text{if at least one } x_i \in [-\pi, 0] \\
& (i = 1, \dots, m).
\end{cases}$$

We extend the function F 2π -periodically in each variable to the whole space $\mathbb{R}^n$.

In [3] we have proved that $F \in H(\omega_k; \mathbb{C}(T^n))$ and

$$\begin{aligned}
& |\Delta_m^k(-h) \tilde{F}_{\{1, \dots, m\}}(0, \dots, 0, \dots, 0)| \\
& \geq D_4 \int_{[0, \pi]^m} \min(h^k, s_m^k) s_m^{-k} \min_{1 \leq i \leq m} \omega_k(s_i) \prod_{i=1}^m s_i^{-1} ds_i,
\end{aligned}$$

where D_4 is a positive constant.

Using the fact that ω_k satisfies Zygmund's condition we get

$$\begin{aligned}
& |\Delta_m^k(-h) \tilde{F}_{\{1, \dots, m\}}(0, \dots, 0, \dots, 0)| \\
& \geq D_5 \omega_k(\tau_p^* - \tau_p) |\ln(\tau_p^* - \tau_p)|^{m-1}, \quad D_5 = \text{const}.
\end{aligned}$$

The inequality (3) is proved. $\square$

Corollary 1. *Let the modulus of continuity ω satisfies Zygmund's condition. Then for each $B \subset \{1, \dots, n\}$ there exist a function $f \in H(\omega, \mathbb{C}(T^n))$ and C, δ_0 positive constants for which we have*

$$\sup_{\bar{h} \in I, |h| \leq \delta} \sup_{\bar{x} \in I} |\tilde{f}_B(\bar{x} + \bar{h}) - \tilde{f}_B(\bar{x})| \geq \omega(\delta) |\ln \delta|^{|B|},$$

where $0 \leq \delta \leq \delta_0$, $B \neq \{1, 2, \dots, n\}$.

$$\sup_{\bar{h} \in I, |h| \leq \delta} \sup_{\bar{x} \in I} |\tilde{f}_B(\bar{x} + \bar{h}) - \tilde{f}_B(\bar{x})| \geq \omega(\delta) |\ln \delta|^{n-1},$$

where $0 \leq \delta \leq \delta_0$, $B = \{1, 2, \dots, n\}$.

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