

SECOND ORDER PARALLEL TENSORS ON LP-SASAKIAN MANIFOLDS WITH A COEFFICIENT α

LOVEJOY S. DAS

ABSTRACT. In 1926, Levy [3] had proved that a second order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [4] has proved that a second order parallel tensor in a Kähler space of constant holomorphic sectional curvature is a linear combination with constant coefficient of the Kählerian metric and the fundamental 2-form. In this paper, we have shown that a second order symmetric parallel tensor on Lorentzian Para Sasakian manifold (briefly LP-Sasakian) with a coefficient α (non zero Scalar function) is a constant multiple of the associated metric tensor and we have also proved that there is no non zero skew symmetric second order parallel tensor on a LP-Sasakian manifold.

1. INTRODUCTION

In 1923, Eisenhart [2] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric tensor is reducible. In 1926 Levy [3] obtained the necessary and sufficient conditions for the existence of such tensors. Sharma [4] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non-singular) tensor on an n -dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [4] that on a Sasakian manifold, there is no non zero parallel 2-form. In this paper we have defined LP-Sasakian manifold with a coefficient α , (non zero scalar function) and have proved the following two theorems:

Theorem 1.1. *On a LP- Sasakian manifold with a coefficient α , a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

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Theorem 1.2. *On a LP-Sasakian manifold with a coefficient α , there is no non zero parallel 2-form.*

Let M be an n -dimensional differentiable manifold of class c^∞ endowed with (1,1) tensor field Φ , a contravariant vector field T , a covariant vector field A and a Lorentzian metric g on M which makes T a timelike unit vector field such that the following conditions are satisfied [1].

$$\begin{aligned}
 (1.1) \quad & A(T) = -1 \\
 (1.2) \quad & \Phi(T) = 0 \\
 (1.3) \quad & A(\Phi X) = 0 \\
 (1.4) \quad & \Phi^2 X = X + A(X)T \\
 (1.5) \quad & A(X) = g(X, T) \\
 (1.6) \quad & g(\Phi X, \Phi Y) = g(X, Y) + A(X)A(Y) \\
 (1.7) \quad & \Phi(X, Y) = g(X, \Phi Y) = g(Y, \Phi X) = \Phi(X, Y) \\
 (1.8) \quad & \Phi(X, T) = 0.
 \end{aligned}$$

Then a manifold satisfying conditions (1.1)–(1.8) is called a LP-Sasakian structure (Φ, T, A, g) on M .

Definition 1.1. If in a LP-Sasakian manifold, the following relation

$$\begin{aligned}
 (1.9) \quad & \Phi X = \frac{1}{\alpha}(\nabla_X T) \\
 (1.10) \quad & \Phi(X, Y) = \frac{1}{\alpha}(\nabla_X A(Y)) = \frac{1}{\alpha}(\nabla_X A)(Y) \\
 (1.11) \quad & \alpha(X) = \nabla_X \alpha \\
 (1.12) \quad & g(X, \bar{\alpha}) = \alpha(X)
 \end{aligned}$$

$$\begin{aligned}
 (1.13) \quad & \nabla_X \Phi(Y, Z) = \\
 & \alpha[\{g(X, Y) + \eta(Y)\eta(X)\}\eta(Z) + \{g(X, Z) + \eta(Z)\eta(X)\}\eta(Y)].
 \end{aligned}$$

hold, where ∇ denotes the Riemannian connection of the metric tensor g , then M is called a LP-Sasakian manifold with coefficient α .

2. PROOFS OF THEOREM 1.1 AND 1.2

In proving Theorems 1.1 and 1.2 we need the following theorems.

Theorem 2.1. *On a LP-Sasakian manifold with coefficient α the following holds*

$$\begin{aligned}
 (2.1) \quad & A(R(X, Y)Z) = \alpha^2[g(Y, Z)A(X) - g(X, Z)A(Y)] \\
 & - [\alpha(X)\Phi(Y, Z) - \alpha(Y)\Phi(X, Z)]
 \end{aligned}$$

Proof. On differentiating (1.10) covariantly and using (1.11), (1.12) and (1.13) the proof follows immediately. $\square$

Theorem 2.2. *For a LP-Sasakian manifold with coefficient α , we have:*

$$(2.2) \quad R(T, X)Y = \alpha^2 [A(Y)X + g(X, Y)T] + \alpha(Y)\Phi X - \bar{\alpha}\Phi(X, Y),$$

where $g(X, \bar{\alpha}) = \alpha(X)$.

Proof. The proof follows in an obvious manner after making use of (1.12) and (2.1). $\square$

Theorem 2.3. *For a LP-Sasakian manifold, with a coefficient α the following holds:*

$$(2.3) \quad R(T, X)T = \beta\phi x + \alpha^2[X + A(X)T]$$

Proof. In view of equation (3.2), the proof follows immediately. $\square$

Proof of Theorem 1.1. Let J denote a $(0, 2)$ -tensor field on a LP-Sasakian manifold M with a coefficient α such that $\nabla J = 0$, then it follows that

$$(2.4) \quad J(R(W, X)Y, Z) + J(Y, R(W, X)Z) = 0$$

holds for arbitrary vector fields X, Y, Z, W on M . Substituting $W = Y = Z = T$ in (2.4) we get

$$(2.5) \quad J(R(T, X)T, T) + J(T, R(T, X)T) = 0.$$

On using Theorem 3.3, the equation (2.5) becomes

$$(2.6) \quad 2\beta J(\Phi X, T) + 2\alpha^2 J(X, T) + 2\alpha^2 g(X, T) J(T, T) = 0.$$

On simplifying (2.6), we get

$$(2.7) \quad -g(X, T) J(T, T) - J(X, T) - \frac{\beta}{\alpha^2} J(\Phi X, T) = 0$$

Replacing X by ΦY in (2.7) we get

$$(2.8) \quad J(\Phi Y, T) = g(\Phi Y, T) J(T, T) + \frac{\beta}{\alpha^2} J(\Phi^2 Y, T)$$

Using (1.4) and (1.5) in the above equation we get

$$(2.9) \quad J(\Phi Y, T) = -\frac{\beta}{\alpha^2} [J(T, T) A(Y) + J(Y, T)]$$

Using (2.7) and (2.9) we get

$$(2.10) \quad J(T, T) A(Y) + J(Y, T) = 0 \text{ if } \alpha^4 + \beta^2 \neq 0$$

Differentiating (2.10) covariantly with respect to y we get

$$(2.11) \quad J(T, T) g(X, \Phi Y) + 2g(X, T) J(\Phi Y, T) + J(X, \Phi Y) = 0$$

From the above equation and (1.9) we obtain

$$(2.12) \quad J(T, T) g(X, \Phi Y) = -J(X, \Phi Y)$$

Replacing Φy by y in (2.12) we get

$$(2.13) \quad J(X, Y) = -J(T, T) g(X, Y)$$

In view of the fact that $J(T, T)$ is constant which can be checked by differentiating it along any vector field on M . Thus we have proved the theorem. $\square$

Proof of Theorem 1.2. Let J be a parallel 2-form on a LP-Sasakian manifold M with a coefficient α . Then putting $W = Y = T$ in (2.4) and using Theorem 3.3 and equations (1.1)–(1.6) we get

$$(2.14) \quad \beta J(\Phi X, Z) + \alpha^2 J(X, Z) + \alpha^2 J(T, Z) A(X) + \alpha^2 J(T, X) A(Z) \\ + J(T, \Phi X) \alpha(Z) - J(\bar{\alpha}, T) \Phi(X, Z) = 0$$

Let us define Φ^* to be a $(2, 0)$ tensor field metrically equivalent to Φ then contracting (2.14) with Φ^* and using the antisymmetry property of J and the symmetry property of Φ^* , we obtain in view of equations (1.3)–(1.6) and after simplifying the following:

$$(2.15) \quad J(\bar{\alpha}, T) = 0.$$

Substituting (2.15) in (2.14) we get

$$(2.16) \quad \beta J(\Phi X, Z) + \alpha^2 [J(X, Z) + J(T, Z) A(X) + J(T, X) A(Z)] \\ + J(T, \Phi X) \alpha(Z) = 0.$$

On simplifying (2.16) we get

$$(2.17) \quad \beta J(\Phi Z, X) + \alpha^2 [J(Z, X) + J(T, X) A(Z) + J(T, Z) A(X)] \\ + J(T, \Phi Z) \alpha(X) = 0.$$

On simplifying (2.16) and (2.17) we get

$$(2.18) \quad -\beta[J(Z, \Phi X) + J(X, \Phi Z)] - \alpha(X) J(\Phi Z, T) - \alpha(Z) J(\Phi X, T) = 0.$$

On replacing X by ΦY in (2.18) we get

$$(2.19) \quad -\beta[J(Z, \Phi^2 Y) + J(\Phi Y, \Phi Z)] - \\ \alpha(\Phi Y) J(\Phi Z, T) - \alpha(Z) J(\Phi^2 Y, T) = 0.$$

On making use of (1.4) in the above equation, we get the following equation:

$$(2.20) \quad -\beta[J(Z, Y) + J(Z, T) A(Y) + J(\Phi Y, \Phi Z)] - \alpha(Z) J(Y, T) \\ - \alpha(\Phi Y) J(\Phi Z, T) = 0.$$

On simplifying (2.20) we get

$$(2.21) \quad -\beta[J(Y, Z) + J(Y, T) A(Z) + J(\Phi Z, \Phi Y)] - \alpha(Y) J(Z, T) \\ - \alpha(\Phi Z) J(\Phi Y, T) = 0.$$

In view of (2.20) and (2.21) and after simplifying we obtain

$$(2.22) \quad \beta[J(T, Z) A(Y) + J(T, Y) A(Z)] + \alpha(Z) J(T, Y) \\ + J(T, \Phi Z) \alpha(\Phi Y) + \alpha(Y) J(Z, T) + \alpha(\Phi Z) J(T, \Phi Y) = 0.$$

Putting $Y = \bar{\alpha}$ in (2.22) and using (2.15) we get

$$(2.23) \quad \beta J(T, Z) A(\bar{\alpha}) + J(T, \Phi Z) \alpha(\Phi \bar{\alpha}) + \alpha(\bar{\alpha}) J(Z, T) = 0$$

Let us put $\alpha \bar{\alpha} = \hat{\alpha}$ and $\hat{\beta} = \alpha(\Phi, \bar{\alpha})$ in (2.23) we get

$$(2.24) \quad J(Z, T) [\beta A(\bar{\alpha}) - \alpha(\bar{\alpha})] = J(T, \Phi Z) \hat{\beta}.$$

Replacing Z by ΦZ in (2.24) we get

$$(2.25) \quad J(\Phi Z, T) [\beta^2 - \bar{\alpha}] = \hat{\beta} J(T, Z).$$

Replacing Z by ΦZ in (2.25) we get

$$(2.26) \quad J(\Phi^2 Z, T) = \frac{\hat{\beta}}{\bar{\alpha} - \beta^2} J(\Phi Z, T).$$

On making use of (2.25) and (1.4) in (2.26) we get

$$(2.27) \quad \frac{\bar{\alpha} - \beta^2}{\hat{\beta}} J(Z, T) = \frac{\hat{\beta}}{\bar{\alpha} - \beta^2} J(Z, T).$$

From (2.27) it follows immediately that

$$(2.28) \quad J(Z, T) = 0 \text{ unless } (\bar{\alpha} - \beta^2)^2 - (\hat{\beta})^2 \neq 0.$$

Using (2.28) in (2.28) we get

$$(2.29) \quad \beta J(Z, \Phi X) + \alpha^2 J(Z, X) = 0$$

Differentiating (2.28) covariantly along Y and using the fact that $\nabla J = 0$ we get

$$(2.30) \quad J(Z, \Phi Y) = 0.$$

In view of (2.30) and (2.29), we see that $J(Y, Z) = 0$. □

REFERENCES

- [1] L. S. Das and J. Sengupta. On conformally flat LP-Sasakian manifolds with a coefficient α . *Bull. Calcutta Math. Soc.*, 98(4):377–382, 2006.
- [2] L. P. Eisenhart. Symmetric tensors of the second order whose first covariant derivatives are zero. *Trans. Amer. Math. Soc.*, 25(2):297–306, 1923.
- [3] H. Levy. Symmetric tensors of the second order whose covariant derivatives vanish. *Ann. of Math. (2)*, 27(2):91–98, 1925.
- [4] R. Sharma. Second order parallel tensors on contact manifolds. *Algebras Groups Geom.*, 7(2):145–152, 1990.

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DEPARTMENT OF MATHEMATICS,
KENT STATE UNIVERSITY,
NEW PHILADELPHIA, OHIO 44663, USA
E-mail address: ldas@kent.edu