

ON A TYPE OF TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. The object of the present paper is to study 3-dimensional trans-Sasakian manifolds admitting a W_2 -curvature tensor. Trans-Sasakian manifolds satisfying the curvature condition $S(X, \xi).R = 0$ is also considered.

1. INTRODUCTION

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinea and Gonzales [5] and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [8], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [12] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4 . The class $C_6 \oplus C_5$ ([10],[11]) coincides with the class of trans-Sasakian structures of type (α, β) . In [11], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [4], some curvature identities and sectional curvatures for C_5 , C_6 and trans-Sasakian manifolds are obtained. It is known that [17] trans-Sasakian structures of type $(0,0)$, $(0,\beta)$, and $(\alpha,0)$ are cosymplectic, β -Kenmotsu and α -Sasakian respectively where $\alpha, \beta \in \mathbb{R}$.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [10]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α -Sasakian or β -Kenmotsu manifold. Hence a proper trans-Sasakian manifold exists only for three dimension. Three-dimensional trans-Sasakian manifolds have been studied by De and Tripathi [7], De and Sarkar [6], Shukla and Singh [15], and many others.

On the other hand, Pokhariyal and Mishra [14] have introduced new tensor fields, called W_2 and E -tensor fields, in a Riemannian manifold, and studied

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their properties. Then, Pokhariyal [13] has studied some properties of this tensor fields in a Sasakian manifold. Recently, De and Sarkar [10] have studied P -Sasakian manifolds admitting W_2 tensor field.

The curvature tensor W_2 is defined by

$$(1.1) \quad W_2(X, Y, U, V) = R(X, Y, U, V) + \frac{1}{n-1}[g(X, U)S(Y, V) - g(Y, U)S(X, V)],$$

where S is a Ricci tensor of type (0,2). The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [19]. According to them a quasi-conformal curvature tensor is defined by

$$(1.2) \quad \tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y, Z)X - g(X, Z)Y],$$

where a and b are non-zero constants, R is the curvature tensor, S is the Ricci tensor, Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$ and r is the scalar curvature of the Riemannian manifold $(M^n, g)(n \geq 3)$. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.2) takes the form

$$(1.3) \quad \tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z,$$

where C is the conformal curvature tensor [18].

On the other hand, the concircular curvature tensor $\tilde{Z}$ in a Riemannian manifold is defined by

$$(1.4) \quad \tilde{Z}(X, Y)U = R(X, Y)U - \frac{r}{n(n-1)}(g(Y, U)X - g(X, U)Y).$$

Again an trans-Sasakian manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant.

The paper is organized as follows: In section 2, some preliminary results are recalled. After preliminaries in section 3, we construct some examples of 3-dimensional trans-Sasakian manifold. Then we have studied a 3-dimensional trans-Sasakian manifold satisfying $W_2 = 0$. In the next section, we have studied W_2 -semisymmetric 3-dimensional trans-Sasakian manifolds. Also, we have classified 3-dimensional trans-Sasakian manifolds satisfying $\tilde{Z}.W_2 = 0$ and $C.W_2 = 0$. Finally we prove that a 3-dimensional trans-Sasakian manifold satisfying the condition $S(X, \xi).R = 0$ is an Einstein manifold, provided $\alpha, \beta = \text{constant}$.

2. PRELIMINARIES

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) , that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$(2.1) \quad \phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

$$(2.3) \quad g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X)$$

for all X and Y tangent to M ([1], [2]).

The fundamental 2-form Φ of the manifold is defined by

$$(2.4) \quad \Phi(X, Y) = g(X, \phi Y)$$

for all X and Y tangent to M .

An almost contact metric structure (ϕ, ξ, η, g) on a connected manifold M is called a trans-Sasakian structure [12] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [8], where J is the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, for any vector fields X on M , f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [3]

$$(2.5) \quad (\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for smooth functions α and β on M . Hence we say that the trans-Sasakian structure is of type (α, β) . From (2.5) it follows that

$$(2.6) \quad \nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi),$$

$$(2.7) \quad (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

An explicit example of a 3-dimensional proper trans-Sasakian manifold is constructed in [10]. In [7], Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given.

From [7] we know that for a 3-dimensional trans-Sasakian manifold

$$(2.8) \quad 2\alpha\beta + \xi\alpha = 0,$$

$$(2.9) \quad S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha,$$

$$(2.10) \quad S(X, Y) = \left(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\right)g(X, Y) \\ - \left(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right)\eta(X)\eta(Y) \\ - (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y),$$

$$(2.11) \quad R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) - \eta(Y)(X\beta)\xi + \phi(X)\alpha\xi \\ + \eta(X)(Y\beta)\xi + \phi(Y)\alpha\xi - (Y\beta)X + (X\beta)Y - (\phi(Y)\alpha)X + (\phi(X)\alpha)Y,$$

and

$$(2.12) \quad R(X, Y)Z = \left(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\right)(g(Y, Z)X - g(X, Z)Y \\ - g(Y, Z)\left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right)\eta(X)\xi \right. \\ \left. - \eta(X)(\phi \operatorname{grad} \alpha - \operatorname{grad} \beta) + (X\beta + (\phi X)\alpha)\xi\right] \\ + g(X, Z)\left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right)\eta(Y)\xi \right. \\ \left. - \eta(Y)(\phi \operatorname{grad} \alpha - \operatorname{grad} \beta) + (Y\beta + (\phi Y)\alpha)\xi\right] \\ - [(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ + \frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)]\eta(Y)\eta(Z)]X \\ + [(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ + \frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)]\eta(X)\eta(Z)]Y,$$

where S is the Ricci tensor of type $(0, 2)$ and R is the curvature tensor of type $(1, 3)$ and r is the scalar curvature of the manifold M .

In a 3-dimensional trans-Sasakian manifold, using (2.9), (2.11) and (2.13), equation (1.3) and (1.4) reduce to

$$(2.13) \quad \tilde{Z}(\xi, X)Y = (\alpha^2 - \beta^2 - \frac{r}{6})\{g(X, Y)\xi - \eta(Y)X\},$$

$$(2.14) \quad C(\xi, Y)W = \frac{(\alpha^2 - \beta^2)(n-1)(n-4) + r}{(n-1)(n-2)}\{g(Y, W)\xi - \eta(W)Y\} \\ - \frac{1}{(n-2)}\{S(Y, W)\xi - \eta(W)QY\},$$

respectively.

3. EXAMPLES OF 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLD

Example 3.1. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinates of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M .

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g , we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$, the set of all smooth vector fields on M .

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M .

Let ∇ be the Levi-Civita connection with respect to the metric g and R be the curvature tensor of M . Then we have

$$[e_1, e_2] = ye_2 - z^2 e_3, \quad [e_1, e_3] = -\frac{1}{z}e_1 \quad \text{and} \quad [e_2, e_3] = -\frac{1}{z}e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g , we can easily calculate

$$\begin{aligned} \nabla_{e_1} e_3 &= -\frac{1}{z}e_1 + \frac{1}{z^2}e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2}z^2 e_3, \quad \nabla_{e_1} e_1 = \frac{1}{z}e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{z}e_2 - \frac{1}{2}z^2 e_1, \\ \nabla_{e_2} e_2 &= ye_1 + \frac{1}{z}e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}z^2 e_3 - ye_2, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = -\frac{1}{2}z^2 e_1, \\ \nabla_{e_3} e_1 &= \frac{1}{2}z^2 e_2. \end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a trans-Sasakian structure on M . Consequently $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold with $\alpha = -\frac{1}{2}z^2 \neq 0$ and $\beta = -\frac{1}{z} \neq 0$.

Example 3.2. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq 0\}$, where (x, y, z) are standard co-ordinates of $\mathbb{R}^3$.

The vector fields

$$e_1 = \frac{\partial}{\partial z} - y \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = 2 \frac{\partial}{\partial x}$$

are linearly independent at each point of M . Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_3) &= g(e_1, e_2) = g(e_2, e_3) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g , we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M .

Let ∇ be the Levi-Civita connection with respect to the metric g . Then we have

$$[e_1, e_2] = e_1 e_2 - e_2 e_1 = \left(\frac{\partial}{\partial z} - y \frac{\partial}{\partial x}\right) \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \left(\frac{\partial}{\partial z} - y \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} = \frac{1}{2} e_3.$$

Similarly,

$$[e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = 0.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g , we can easily calculate

$$\begin{aligned} \nabla_{e_1} e_3 &= \frac{1}{4} e_2, & \nabla_{e_1} e_2 &= -\frac{1}{4} e_3, & \nabla_{e_1} e_1 &= 0, \\ \nabla_{e_2} e_3 &= -\frac{1}{4} e_1, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_1 &= \frac{1}{4} e_3, \\ \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 &= -\frac{1}{4} e_1, & \nabla_{e_3} e_1 &= \frac{1}{4} e_2. \end{aligned}$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula (2.6) for $\alpha = \frac{1}{4}$ and $\beta = 0$. Hence the manifold is a trans-Sasakian manifold of type $(\frac{1}{4}, 0)$.

4. 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS SATISFYING $W_2 = 0$

In this section we consider a 3-dimensional trans-Sasakian manifolds satisfying $W_2 = 0$. Then we have from (1.1)

$$(4.1) \quad R(X, Y, U, V) = \frac{1}{n-1} [g(Y, U)S(X, V) - g(X, U)S(Y, V)].$$

Using $X = U = \xi$ in (4.1), we have

$$(4.2) \quad R(\xi, Y, \xi, V) = \frac{1}{n-1} [g(Y, \xi)S(\xi, V) - g(\xi, \xi)S(Y, V)].$$

From (2.1), (2.9) and (2.11), we get

$$(4.3) \quad S(Y, V) = 2(\alpha^2 - \beta^2 - \xi\beta)g(Y, V) + (\xi\beta)\eta(Y)\eta(V) - \{(\phi V)\alpha\}\eta(Y) - (V\beta)\eta(Y).$$

If α and β are constant, then we have

$$(4.4) \quad S(Y, V) = 2(\alpha^2 - \beta^2)g(Y, V).$$

Thus we have the following:

Theorem 4.1. *A 3-dimensional trans-Sasakian manifold satisfying $W_2 = 0$ is an Einstein manifold, provided $\alpha, \beta = \text{constant}$.*

Now using (4.4) in (4.1), we get

$$(4.5) \quad R(X, Y, U, V) = (\alpha^2 - \beta^2)[g(Y, U)g(X, V) - g(X, U)g(Y, V)].$$

Corollary 4.1. *A 3-dimensional trans-Sasakian manifold satisfying $W_2 = 0$ is a manifold of constant curvature $(\alpha^2 - \beta^2)$, provided $\alpha, \beta = \text{constant}$.*

5. W_2 -SEMISSYMMETRIC 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS

A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([16],[9]) if $R(X, Y).R = 0$, where R is the Riemannian curvature tensor and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y . If a Riemannian manifold satisfies

$$(5.1) \quad R(X, Y).W_2 = 0,$$

then the manifold is said to be W_2 semi-symmetric manifold.

Proposition 5.1. *Let M be an 3-dimensional trans-Sasakian manifold. Then the W_2 -curvature tensor on M satisfies the condition*

$$(5.2) \quad W_2(X, Y, U, \xi) = 0.$$

From (5.1) we have

$$(5.3) \quad R(X, Y)W_2(Z, U)V - W_2(R(X, Y)Z, U)V \\ - W_2(Z, R(X, Y)U)V - W_2(Z, U)R(X, Y)V = 0.$$

This equation implies

$$(5.4) \quad g(R(X, Y)W_2(Z, U)V, \xi) - g(W_2(R(X, Y)Z, U)V, \xi) \\ - g(W_2(Z, R(X, Y)U)V, \xi) - g(W_2(Z, U)R(X, Y)V, \xi) = 0.$$

Putting $X = \xi$ in (5.4) we obtain

$$(5.5) \quad g(R(\xi, Y)W_2(Z, U)V, \xi) - g(W_2(R(\xi, Y)Z, U)V, \xi) \\ - g(W_2(Z, R(\xi, Y)U)V, \xi) - g(W_2(Z, U)R(\xi, Y)V, \xi) = 0.$$

Using (5.4) in (5.5), we get

$$(5.6) \quad -g(Y, W_2(Z, U)V)\xi + \eta(W_2(Z, U)V)Y + g(Y, Z)g(W_2(\xi, U)V, \xi) \\ - \eta(Z)g(W_2(Y, U)V, \xi) + g(Y, U)g(W_2(Z, \xi)V, \xi) - \eta(U)g(W_2(Z, Y)V, \xi) \\ + g(Y, V)g(W_2(Z, U)\xi, \xi) - \eta(V)g(W_2(Z, U)Y, \xi) = 0.$$

Taking the inner product with ξ and using (5.2) in (5.7), we obtain

$$W_2(Z, U, V, Y) = 0.$$

Then from previous Theorem and Corollary we have

Theorem 5.1. *A W_2 -semisymmetric 3-dimensional trans-Sasakian manifold is an Einstein manifold and hence a manifold of constant curvature $(\alpha^2 - \beta^2)$, provided $\alpha, \beta = \text{constant}$.*

6. 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS SATISFYING
 $\tilde{Z}(X, Y).W_2 = 0$

In this section we consider a 3-dimensional trans-Sasakian manifolds satisfying the condition

$$(6.1) \quad \tilde{Z}(X, Y).W_2 = 0.$$

This equation implies

$$(6.2) \quad \begin{aligned} \tilde{Z}(X, Y)W_2(Z, U)V - W_2(\tilde{Z}(X, Y)Z, U)V \\ - W_2(Z, \tilde{Z}(X, Y)U)V - W_2(Z, U)\tilde{Z}(X, Y)V = 0. \end{aligned}$$

Putting $X = \xi$ in (6.2) we obtain

$$(6.3) \quad \begin{aligned} \tilde{Z}(\xi, Y)W_2(Z, U)V - W_2(\tilde{Z}(\xi, Y)Z, U)V \\ - W_2(Z, \tilde{Z}(\xi, Y)U)V - W_2(Z, U)\tilde{Z}(\xi, Y)V = 0. \end{aligned}$$

Using (2.13) in (6.3), we obtain

$$(6.4) \quad \begin{aligned} (\alpha^2 - \beta^2 - \frac{r}{6})\{g(Y, W_2(Z, U)V)\xi - g(W_2(Z, U)V, \xi)Y \\ - g(Y, Z)W_2(\xi, U)V + \eta(Z)W_2(Y, U)V - g(Y, U)W_2(Z, \xi)V \\ \eta(U)W_2(Z, U)V - g(Y, V)W_2(Z, U)\xi + \eta(V)W_2(Z, U)Y\} = 0. \end{aligned}$$

Taking the inner product with ξ and using (4.2) in (6.5), we have

$$(6.5) \quad (\alpha^2 - \beta^2 - \frac{r}{6})g(Y, W_2(Z, U)V) = 0.$$

Again from (2.13) we have $(\alpha^2 - \beta^2 - \frac{r}{6}) \neq 0$. Hence we have

$$(6.6) \quad W_2(Z, U, V, Y) = 0.$$

From the proof of Theorem 4.1 and Corollary 4.1 we have

Theorem 6.1. *A 3-dimensional trans-Sasakian manifold satisfying the condition $\tilde{Z}(X, Y).W_2 = 0$ is an Einstein manifold and hence a manifold of constant curvature $(\alpha^2 - \beta^2)$, provided $\alpha, \beta = \text{constant}$.*

7. 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS SATISFYING
 $C(X, Y).W_2 = 0$

In this section we characterize the 3-dimensional trans-Sasakian manifold satisfying the condition

$$(7.1) \quad C(X, Y).W_2 = 0.$$

This equation implies

$$(7.2) \quad \begin{aligned} C(X, Y)W_2(Z, U)V - W_2(C(X, Y)Z, U)V \\ - W_2(Z, C(X, Y)U)V - W_2(Z, U)C(X, Y)V = 0. \end{aligned}$$

Putting $X = \xi$ in (7.2) we obtain

$$(7.3) \quad C(\xi, Y)W_2(Z, U)V - W_2(C(\xi, Y)Z, U)V \\ - W_2(Z, C(\xi, Y)U)V - W_2(Z, U)C(\xi, Y)V = 0.$$

Using (2.14) in (7.3), we obtain

$$(7.4) \quad \left(\frac{(\alpha^2 - \beta^2)(n-1)(n-4) + r}{(n-1)(n-2)} \right) \{ g(Y, W_2(Z, U)V)\xi - g(W_2(Z, U)V, \xi)Y \\ - g(Y, Z)W_2(\xi, U)V + \eta(Z)W_2(Y, U)V - g(Y, U)W_2(Z, \xi)V \\ \eta(U)W_2(Z, U)V - g(Y, V)W_2(Z, U)\xi + \eta(V)W_2(Z, U)Y \} = 0.$$

Taking the inner product with ξ and using (4.2) in (7.5), we have

$$(7.5) \quad \left(\frac{(\alpha^2 - \beta^2)(n-1)(n-4) + r}{(n-1)(n-2)} \right) g(Y, W_2(Z, U)V) = 0.$$

Let U_1 and U_2 be a part of M satisfying $(\alpha^2 - \beta^2)(n-1)(n-4) + r = 0$ and

$$(7.6) \quad W_2(Z, U, V, Y) = 0.$$

This leads to the following:

Theorem 7.1. *Let M be a 3-dimensional trans-Sasakian manifold satisfying the condition $C(X, Y).W_2 = 0$. Then either $(\alpha^2 - \beta^2)(n-1)(n-4) + r = 0$, or M is a manifold of constant curvature $(\alpha^2 - \beta^2)$, provided $\alpha, \beta = \text{constant}$.*

8. 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS SATISFYING

$$S(X, \xi).R = 0$$

We now consider a 3-dimensional trans-Sasakian manifold satisfying the condition

$$(8.1) \quad S(X, \xi).R = 0.$$

By definition we have

$$(8.2) \quad (S(X, \xi).R)(U, V)Z = ((X \wedge_S \xi).R)(U, V)Z \\ = (X \wedge_S \xi)R(U, V)Z + R((X \wedge_S \xi)U, V)Z \\ + R(U, (X \wedge_S \xi)V)Z + R(U, V)(X \wedge_S \xi)Z,$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(8.3) \quad (X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y.$$

Using the definition of (8.3) in (8.2) we get by virtue of (8.1)

$$(8.4) \quad S(\xi, R(U, V)Z)X - S(X, R(U, V)Z)\xi + R(S(\xi, U)X - S(X, U)\xi, V)Z \\ + R(U, S(\xi, V)X - S(X, V)\xi)Z + R(U, V)\{S(\xi, Z)X - S(X, Z)\xi\} = 0.$$

Taking the inner product of (8.4) by ξ we obtain

$$(8.5) \quad S(\xi, R(U, V)Z)\eta(X) - S(X, R(U, V)Z) + S(\xi, U)\eta R(X, V)Z$$

$$\begin{aligned}
& -S(X, U)\eta(R(\xi, V)Z) + S(\xi, V)\eta(R(U, X)Z - S(X, V)\eta(R(U, \xi)Z) \\
& + S(\xi, Z)\eta(R(U, V)X) - S(X, Z)\eta(R(U, V)\xi) = 0.
\end{aligned}$$

Putting $U = Z = \xi$ in (8.5) and using (2.9) and (2.11) we get

$$(8.6) \quad S(X, V) = 2(\alpha^2 - \beta^2)^2 g(X, V) + 4(\alpha^2 - \beta^2)^2 \eta(X)\eta(V),$$

provided $\alpha, \beta = \text{constant}$. This leads to the following:

Theorem 8.1. *3-dimensional trans-Sasakian manifold satisfying the condition $S(X, \xi) \cdot R = 0$ is an Einstein manifold, provided $\alpha, \beta = \text{constant}$.*

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