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THE STRUCTURE OF THE UNIT GROUP OF THE GROUP ALGEBRA $\mathbb{F}S_5$ WHERE $\mathbb{F}$ IS A FINITE FIELD WITH $\mathbf{char}(\mathbb{F}) = p > 5$

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Abstract. Let $\mathbb{F}_q$ denote a field having $q = p^n$ elements, where p a prime, and let S_5 denote the symmetric group of degree 5. We give a complete structure of the unit group $\mathcal{U}(\mathbb{F}_q S_5)$, $p > 5$.

1. Introduction

Determination of unit group of a group algebra has been very fascinating and challenging. Recently, Gildea [2, 3, 4, 5, 6, 7, 8, 9, 10, 12] and Makhijani [10] have determined the unit group of certain finite group algebras. Makhijani [10] characterized the unit group of the finite group algebra of the alternating group A_5. In this paper we have characterized completely for $\mathbb{F}S_5$, $\operatorname{char}\mathbb{F} > 5$.

Let $\mathbb{F}$ be a finite field of $\operatorname{char}(\mathbb{F}) = p > 5$. We shall write $\mathbb{F} = \mathbb{F}_q = \mathbb{F}_{p^n}$, where $|\mathbb{F}| = p^n$. Since $|S_5| = 120 = 2^3 \cdot 3 \cdot 5 \cdot$ The Jacobson radical $J(\mathbb{F}_q S_5) = 0$ by Maschke's theorem. Also $\mathbb{F}S_5$ is a finite ring, hence Artinian. Further, by Wedderburn Decomposition theorem

$$\mathbb{F}_q S_5 \cong \mathbb{M}_{n_1}(D_1) \oplus \mathbb{M}_{n_2}(D_2) \oplus \cdots \oplus \mathbb{M}_{n_r}(D_r)$$

where $D_1, D_2, \ldots, D_r$ are finite dimensional division algebras over the finite field $\mathbb{F}$. Therefore each $D_k = \mathbb{F}_{p^{n_k}}$ is a finite field and

$$\mathbb{F}_q S_5 \cong \mathbb{M}_{n_1}(\mathbb{F}_{p^{m_1}}) \oplus \mathbb{M}_{n_2}(\mathbb{F}_{p^{m_2}}) \oplus \cdots \oplus \mathbb{M}_{n_r}(\mathbb{F}_{p^{m_r}}).$$

Hence

$$\mathcal{U}(\mathbb{F}_q S_5) \cong GL_{n_1}(\mathbb{F}_{p^{m_1}}) \times GL_{n_2}(\mathbb{F}_{p^{m_2}}) \times \cdots \times GL_{n_r}(\mathbb{F}_{p^{m_r}})$$

Thus the structure of the unit group $\mathcal{U}(\mathbb{F}_q S_5)$ is determined completely once we compute $r, n_1, n_2, \ldots, n_r, m_1, m_2, \ldots, m_r$. Finding r is easy. Clearly the centre,

$$\mathcal{Z}(\mathbb{F}_q S_5) \cong \mathbb{F}_{p^{m_1}} \oplus \mathbb{F}_{p^{m_2}} \oplus \cdots \oplus \mathbb{F}_{p^{m_r}}$$

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where r is the number of distinct conjugacy classes of S_5. It can be seen easily that $r = 7$. We shall use the following presentation of the group S_5

$$S_5 = \langle a, b \mid a^2, b^5, (ab)^4, (bab^{-2}ab)^2\rangle$$

where $a = (1, 2)$ and $b = (1, 2, 3, 4, 5)$.

2. Preliminaries

Through out the paper, $\mathbb{F}$ will denote a finite field with p^n elements and G will denote a finite group. We shall need the following results from Ferraz [1].

An element $x \in G$ is called $p-$regular if $p \nmid |x|$. Let s be the l.c.m. of the orders of the p-regular elements of G and θ be a primitive s^{th} root of unity over $\mathbb{F}$, the multiplicative group $T_{G,\mathbb{F}}$ is defined by

$$T_{G,\mathbb{F}} = \{t \mid \theta \to \theta^t \text{ is an automorphism of } \mathbb{F}(\theta) \text{ over } \mathbb{F}\}.$$

For p-regular elements g, denote by γ_g the sum of all conjugates of g in G. The cyclotomic $\mathbb{F}-$class of γ_g is to be the set

$$S_{\mathbb{F}}(\gamma_g) = \{\gamma_{g^t} | t \in T_{G,\mathbb{F}}\}.$$

Proposition 2.1 ([1])**.** *The number of simple components of $\frac{\mathbb{F}G}{J(\mathbb{F}G)}$ is equal to the number of cyclotomic $\mathbb{F}-$classes in G.*

Proposition 2.2 ([1])**.** *Suppose the Galois group $Gal(\mathbb{F}(\theta) : \mathbb{F})$ is cyclic and t be the number of cyclotomic $\mathbb{F}-$classes in G. If $K_1, K_2, \ldots, K_t$ are the simple components of $\mathcal{Z}(\frac{\mathbb{F}G}{J(\mathbb{F}G)})$ and $S_1, S_2, \ldots, S_t$ are the cyclotomic $\mathbb{F}-$classes of G, then $|S_i| = [K_i : \mathbb{F}]$ with a suitable ordering of the indices.*

From the above proposition it follows that

$$\mathcal{Z}(\frac{\mathbb{F}G}{J(\mathbb{F}G)}) \cong \mathbb{F}_{p^{n_1}} \oplus \mathbb{F}_{p^{n_2}} \oplus \cdots \oplus \mathbb{F}_{p^{n_t}},$$

where $\mathbb{F}_{n_i}$ denotes the unique field between $\mathbb{F}(\theta)$ and $\mathbb{F}$ such that $[\mathbb{F}_{n_i} : \mathbb{F}] = n_i$.

We shall freely use Feeraz's [1] result, since every finite extension of a finite field is a cyclic extension which gives $Gal(\mathbb{F}(\theta) : \mathbb{F})$ cyclic group. Again, $T_{G,\mathbb{F}_q} = \{q^i \; mods \mid 0 \leq i \leq d - 1\}$, where d is the order of $q \mod s$. We shall use the above form of $T_{G,\mathbb{F}}$ to compute order of the cyclotomic $\mathbb{F}$-classes in G, and the relation given in proposition 2.2.

3. The structure of the unit group $\mathcal{U}(\mathbb{F}_q S_5), p > 5$

We need the following result given in [11]

Proposition 3.1 (Prop. 3.6.11, [11])**.** *Let $\mathbb{F}G$ be a semisimple group algebra. If G' denotes the commutator subgroup of G then we can write*

$$\mathbb{F}G \cong \mathbb{F}G_{e_{G'}} \oplus \triangle(G, G'),$$

where $\mathbb{F}G_{e_{G'}} \cong \mathbb{F}(\frac{G}{G'})$ *is the sum of all commutative simple components of* $\mathbb{F}G$ *and* $\triangle(G, G')$ *is the sum of all the others. Here* $e_{G'} = \frac{\widehat{G'}}{|G'|}$ *where* $\widehat{G'}$ *is the sum of all elements of* G'.

Theorem 3.2. *Let* $q = p^n, p > 5$ *be a prime then,* $\mathbb{F}_q S_5$ *is isomorphic to one of the following:*

(i) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_9(\mathbb{F}_q)$,
(ii) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_6(\mathbb{F}_q) \oplus \mathbb{M}_7(\mathbb{F}_q)$,
(iii) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_4(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_8(\mathbb{F}_q)$,
(iv) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_4(\mathbb{F}_q) \oplus \mathbb{M}_4(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_6(\mathbb{F}_q)$.

Proof. In our case $\frac{S_5}{S_5'} \cong C_2$ as $S_5' = A_5$. Also, $|S_5| = 2^3 \cdot 3 \cdot 5$ and therefore group algebra $\mathbb{F}_q S_5$ is semi-simple as $p \nmid |G|$. Here $q = p^n$, $p > 5$.

By proposition 3.1

$$\mathbb{F}_q S_5 = \mathbb{F}_q S_5 e_{S_5'} \oplus \mathbb{F}_q S_5(S_5' - 1)$$

where $e_{S_5'} = e_{A_5} = \frac{\widehat{A_5'}}{|A_5|} = \frac{\sum_{\sigma \in A_5} \sigma}{60}$

$$\mathbb{F}_q S_5 e_{S_5'} = \text{ sum of all commutative simple components of } \mathbb{F}_q S_5.$$

However,

$$\mathbb{F}_q S_5 e_{S_5'} \cong \mathbb{F}_q(\frac{S_5}{S_5'}) \cong \mathbb{F}_q(C_2) \cong \mathbb{F}_q \oplus \mathbb{F}_q.$$

Therefore, by Wedderburn Decomposition Theorem

$$\mathbb{F}_q S_5 \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus \sum_{i=1}^{5} \mathbb{M}_{n_i}(\mathbb{F}_{q^{k_i}}),$$

for $n_i \geq 2$ all are the seven simple components.

Now $p > 5$, so $q = p^n \equiv \pm 1 \mod 6$. Again $|S_{\mathbb{F}_q}(\gamma_g)| = 1$, for each $g \in S_5$. Hence each $k_i = 1$ in the Wedderburn Decomposition of $\mathbb{F}_q S_5$. Since $p > 5$, therefore each element of S_5 is $p-$regular.

By dimension constraints, we have

$$\dim_{\mathbb{F}_q}(\mathbb{F}_q S_5) = 1 + 1 + n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$$

$$120 = 1 + 1 + \sum_{k=1}^{5} n_k^2$$

$$118 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$$

Possible solutions of the equation are

$$2, 2, 2, 5, 9$$

$$2, 2, 5, 6, 7$$

$$2, 3, 4, 5, 8$$

$$4, 4, 5, 5, 6.$$

Hence $\mathbb{F}_q S_5$ is isomorphic to one of the following:

(i) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_9(\mathbb{F}_q)$,
(ii) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_6(\mathbb{F}_q) \oplus \mathbb{M}_7(\mathbb{F}_q)$,
(iii) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_4(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_8(\mathbb{F}_q)$,
(iv) $\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{M}_4(\mathbb{F}_q) \oplus \mathbb{M}_4(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_5(\mathbb{F}_q) \oplus \mathbb{M}_6(\mathbb{F}_q)$.

□

Corollary 3.3. *Let $q = p^n$, where $p > 5$ be a prime, then $\mathcal{U}(\mathbb{F}_q S_5)$ is isomorphic to one of the following:*

(i) $\mathcal{U}(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_2(\mathbb{F}_q) \times GL_2(\mathbb{F}_q) \times GL_2(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_9(\mathbb{F}_q)$,
(ii) $\mathcal{U}(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_2(\mathbb{F}_q) \times GL_2(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_6(\mathbb{F}_q) \times GL_7(\mathbb{F}_q)$,
(iii) $\mathcal{U}(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_2(\mathbb{F}_q) \times GL_3(\mathbb{F}_q) \times GL_4(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_8(\mathbb{F}_q)$,
(iv) $\mathcal{U}(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_4(\mathbb{F}_q) \times GL_4(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_6(\mathbb{F}_q)$.

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