

ON A HYPERBOLIC KAEHLERIAN SPACE

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ABSTRACT. The object of the present paper is to study some curvature properties in a hyperbolic Kaehlerian manifold equipped with quarter-symmetric metric connection.

1. INTRODUCTION

Hyperbolic Kaehlerian manifold has been studied by different differential geometer through different approaches. Nevena Pušić [5] studied hyperbolic Kaehlerian space equipped with quarter-symmetric metric connection. In 1985, G. Ganchev and A. Borisov [3] discussed the isotropic sections and curvature properties of hyperbolic Kaehlerian manifolds. Nevena Pušić [6] discussed HB-parallel hyperbolic Kaehlerian spaces. In 2013, Arif Salimov and S. Turanli [1] has been discussed some curvature properties of anti-Kaehler-codazzi manifold. Recently, hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection has been studied by B.B. Chaturvedi and B.K. Gupta [2] in 2015. In the consequences of these studies, in this paper we have studied some curvature properties of a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection.

If F_i^h satisfies the relation

$$(1.1) \quad F_j^i F_i^h = \delta_j^h,$$

$$(1.2) \quad F_{ij} = -F_{ji}, \quad (F_{ij} = g_{jk} F_i^k),$$

and

$$(1.3) \quad F_{i,j}^h = 0,$$

then the manifold is called hyperbolic Kaehlerian (space) manifold.

Where F_i^h is a tensor field of type (1,1) and $F_{i,j}^h$ is a covariant derivative of F_i^h with respect to Riemannian connection.

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In 1975 S. Global [7] defined

Definition 1.1. A linear connection ∇ on a n -dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection if the torsion tensor T , defined by

$$(1.4) \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

of the connection ∇ , satisfies

$$(1.5) \quad T(X, Y) = \eta(X)\phi Y - \eta(Y)\phi X,$$

where η is a 1-form and ϕ is a tensor field of type $(1,1)$.

A quarter-symmetric connection ∇ is said to be a quarter-symmetric metric connection if the covariant derivative of metric g vanishes otherwise it is called a quarter-symmetric non-metric connection.

Yano and Imai [4] considered a quarter-symmetric metric connection ∇ and Riemannian connection D with coefficients Γ_{ij}^h and $\{_{ij}^h\}$ respectively. According to them if the torsion tensor T of the connection ∇ on (M^n, g) , $(n > 2)$, satisfies

$$(1.6) \quad T_{jk}^i = p_j A_k^i - p_k A_j^i,$$

then the relation between the coefficients of quarter-symmetric metric connection ∇ and Riemannian connection D is given by

$$(1.7) \quad \Gamma_{jk}^i = \{_{jk}^i\} - p_k U_j^i + p_j V_k^i - p^i V_{jk},$$

where

$$(1.8) \quad U_{ij} = \frac{1}{2}(A_{ij} - A_{ji}), \quad V_{ij} = \frac{1}{2}(A_{ij} + A_{ji}),$$

where $\nabla g = 0$ and p_i are the components of 1-form. Also A_j^i denotes the components of a tensor of type $(1,1)$. U_{ij} and V_{ij} are covariant skew symmetric and symmetric tensors respectively.

Equation (1.8) implies

$$(1.9) \quad A_{ij} = U_{ij} + V_{ij}.$$

Nevena Pušić [6] found a relation between Γ_{ij}^h and $\{_{ij}^h\}$ by putting $V_{ij} = g_{ij}$ and $U_{ij} = F_{ij}$ in (1.7), given by

$$(1.10) \quad \Gamma_{jk}^i = \{_{jk}^i\} - p_k F_j^i + p_j \delta_k^i - p^i g_{jk}.$$

where $\omega^h = \omega_t g^{th}$, ω^h is a contravariant components of generating vector w_h .

Also, Nevena Pušić [6] found a relation between curvature tensor with respect to a quarter-symmetric metric connection ∇ and a Riemannian connection D given by

$$(1.11) \quad \begin{aligned} \bar{R}_{ijkh} = & R_{ijkh} - g_{ih} p_{kj} + g_{ik} p_{hj} - g_{jk} p_{hi} + g_{hj} p_{ki} \\ & + p_j p_h F_{ik} + p_i p_k F_{jh} - p_j p_k F_{ih} - p_i p_h F_{jk}, \end{aligned}$$

where

$$(1.12) \quad p_{jk} = \nabla_j p_k - p_j p_k + p_k q_j + \frac{1}{2} p_s p^s g_{jk}.$$

Taking covariant derivative of F_i^h with respect to quarter-symmetric metric connection ∇ and Riemannian connection D , we have

$$(1.13) \quad \nabla_k F_i^h = \partial_k F_i^h + F_i^r \Gamma_{rk}^h - F_r^h \Gamma_{ik}^r,$$

and

$$(1.14) \quad D_k F_i^h = \partial_k F_i^h + F_i^r \{^h_{rk}\} - F_r^h \{^r_{ik}\}.$$

Subtracting (1.14) from (1.13), we have

$$(1.15) \quad \nabla_k F_i^h - D_k F_i^h = F_i^r (\Gamma_{rk}^h - \{^h_{rk}\}) - F_r^h (\Gamma_{ik}^r - \{^r_{ik}\}).$$

Using (1.10) in (1.15), we have

$$(1.16) \quad \nabla_k F_i^h - D_k F_i^h = F_i^r (-p_k F_r^h + p_r \delta_k^h - p^h g_{rk}) - F_r^h (-p_k F_i^r + p_i \delta_k^r - p^r g_{ik}).$$

Using (1.1) and (1.2) in (1.16), we have

$$(1.17) \quad \nabla_k F_i^h = D_k F_i^h.$$

Again taking covariant derivative of (1.13) with respect to quarter-symmetric metric connection, we get

$$(1.18) \quad \begin{aligned} \nabla_j \nabla_k F_i^h = & \partial_j \partial_k F_i^h - \partial_r F_i^h \Gamma_{jk}^r - \partial_k F_r^h \Gamma_{ij}^r \\ & + \partial_k F_i^r \Gamma_{rj}^h + (\partial_j F_i^r + F_i^m \Gamma_{mj}^r - F_m^r \Gamma_{ij}^m) \Gamma_{rk}^h \\ & + F_i^r \nabla_j \Gamma_{rk}^h - (\partial_j F_r^h + F_r^m \Gamma_{mj}^h - F_m^h \Gamma_{rk}^m) \Gamma_{ik}^r - F_r^h \nabla_j \Gamma_{ik}^r. \end{aligned}$$

Interchanging j and k in equation (1.18), we get

$$(1.19) \quad \begin{aligned} \nabla_k \nabla_j F_i^h = & \partial_k \partial_j F_i^h - \partial_r F_i^h \Gamma_{jk}^r - \partial_j F_r^h \Gamma_{ik}^r \\ & + \partial_j F_i^r \Gamma_{rk}^h + (\partial_k F_i^r + F_i^m \Gamma_{mk}^r - F_m^r \Gamma_{ik}^m) \Gamma_{rj}^h \\ & + F_i^r \nabla_k \Gamma_{rj}^h - (\partial_k F_r^h + F_r^m \Gamma_{mk}^h - F_m^h \Gamma_{rk}^m) \Gamma_{ij}^r - F_r^h \nabla_k \Gamma_{ij}^r. \end{aligned}$$

Subtracting (1.18) from (1.19), we get

$$(1.20) \quad \begin{aligned} \nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h = & F_i^m (\Gamma_{mk}^r \Gamma_{rj}^h - \Gamma_{mj}^r \Gamma_{rk}^h + \nabla_k \Gamma_{mj}^h - \nabla_j \Gamma_{mk}^h) \\ & - F_r^h (\Gamma_{mk}^r \Gamma_{ij}^m - \Gamma_{mj}^r \Gamma_{ik}^m + \nabla_j \Gamma_{ik}^r - \nabla_k \Gamma_{ij}^r). \end{aligned}$$

Equation (1.20) implies

$$(1.21) \quad \nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h = \bar{R}_{kjm}^h F_i^m - \bar{R}_{kji}^r F_r^h.$$

2. TWIN ANTI-HERMITIAN METRIC

A skew symmetric tensor ω defined by

$$(2.1) \quad \omega(Y, Z) = g(FY, Z),$$

is said to be a killing-yano tensor if

$$(2.2) \quad (D_X \omega)(Y, Z) + (D_Y \omega)(X, Z) = 0.$$

The twin anti-Hermitian metric G defined by

$$(2.3) \quad G(Y, Z) = g(FY, Z),$$

where $G(Y, Z) = G(Z, Y)$.

Since G is symmetric but 2-form ω is not symmetric so the killing-yano equation (2.2) has no immediate meaning. Therefore, we can change the killing-yano equation by Codazzi equation

$$(2.4) \quad (D_X G)(Y, Z) - (D_Y G)(X, Z) = 0.$$

Equation (2.4) equivalent to

$$(2.5) \quad (D_X F)Y - (D_Y F)X = 0.$$

3. CURVATURE PROPERTIES

We know that for Riemannian connection D

$$(3.1) \quad D_k D_j F_i^h - D_j D_k F_i^h = R_{kjm}^h F_i^m - R_{kji}^r F_r^h.$$

Now subtracting (1.21) from (3.1), we get

$$(3.2) \quad (D_k D_j F_i^h - \nabla_k \nabla_j F_i^h) - (D_j D_k F_i^h - \nabla_j \nabla_k F_i^h) = R_{kjm}^h F_i^m - R_{kji}^r F_r^h \\ - \bar{R}_{kjm}^h F_i^m + \bar{R}_{kji}^r F_r^h.$$

After contracting (3.2) by h and k and using (1.3) and (1.17), we get

$$(3.3) \quad (D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - R_{hji}^r F_r^h - \bar{S}_{jm} F_i^m + \bar{R}_{hji}^r F_r^h.$$

Equation (3.3) can be written as

$$(3.4) \quad (D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - R_{hji}^r g^{rl} F_r^h - \bar{S}_{jm} F_i^m + \bar{R}_{hji}^r g^{rl} F_r^h.$$

Using $g^{rl} F_r^h = G^{hl}$ in (3.4), we get

$$(3.5) \quad (D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - R_{hji}^r G^{hl} - \bar{S}_{jm} F_i^m + \bar{R}_{hji}^r G^{hl}.$$

In 2013, Arif Salimov and S. Turanli [6] defined

$$(3.6) \quad H_{ji} = R_{hji}^r G^{hl}.$$

Now we are taking

$$(3.7) \quad \bar{H}_{ji} = \bar{R}_{hji}^r G^{hl},$$

using (3.6) and (3.7) in (3.5), we have

$$(3.8) \quad (D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - H_{ji} - \bar{S}_{jm} F_i^m + \bar{H}_{ji},$$

where S_{jm} and $\bar{S}_{jm}$ are Ricci tensors with respect to Riemannian connection and quarter-symmetric metric connection respectively and G^{hl} is twin anti-Hermitian metric.

We know that the curvature tensor of type (0,4) with respect to Riemannian connection D satisfies the following relations

$$(3.9) \quad (i) \quad R_{(hj)il} = 0 \quad \text{and} \quad (ii) \quad R_{hj(il)} = 0.$$

Now, equation (3.6) can be written as

$$(3.10) \quad H_{ji} = \frac{1}{2}(R_{hji l} + R_{lji h}) G^{lh}.$$

Interchanging i and j in (3.10), we get

$$(3.11) \quad H_{ij} = \frac{1}{2}(R_{hij l} + R_{lij h}) G^{lh}.$$

Subtracting (3.11) from (3.10), we have

$$(3.12) \quad H_{ji} - H_{ij} = \frac{1}{2}(R_{hji l} + R_{lji h} - R_{hij l} - R_{lij h}) G^{lh} = 0.$$

Equation (3.12) implies that

$$(3.13) \quad H_{ji} = H_{ij}.$$

If we take

$$(3.14) \quad \begin{aligned} (i) \quad & p_{hi} F_i^h = p_{ih} F_j^h, \\ (ii) \quad & p_j F_{hi} = p_i F_{jh} \\ (iii) \quad & p_i^r g_{hj} = p_j^r g_{hi}, \end{aligned}$$

then from (1.11) and (3.7) we can say that $\bar{R}_{hji l}$ will be symmetric in first and last indices.

Therefore we can write

$$(3.15) \quad \bar{H}_{ji} = \frac{1}{2}(\bar{R}_{hji l} + \bar{R}_{lji h}) G^{lh}.$$

Interchanging i and j in (3.15), we get

$$(3.16) \quad \bar{H}_{ij} = \frac{1}{2}(\bar{R}_{hij l} + \bar{R}_{lij h}) G^{lh}.$$

Subtracting (3.16) from (3.15), we get

$$(3.17) \quad \bar{H}_{ji} - \bar{H}_{ij} = \frac{1}{2}(\bar{R}_{hji l} + \bar{R}_{lji h} - \bar{R}_{hij l} - \bar{R}_{lij h}) G^{lh} = 0,$$

equation (3.17) implies that

$$(3.18) \quad \bar{H}_{ji} = \bar{H}_{ij}.$$

Thus we conclude

Theorem 3.1. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection H_{ij} is symmetric with respect to quarter-symmetric metric connection ∇ if equation (3.14) holds.*

Now equation (3.8) can be written as

$$(3.19) \quad D_h(D_j F_i^h - D_i F_j^h) - \nabla_h(\nabla_j F_i^h - \nabla_i F_j^h) = (S_{jm} F_i^m - H_{ji}) - (S_{im} F_j^m - H_{ij}) + (\bar{S}_{im} F_j^m - \bar{H}_{ij}) - (\bar{S}_{jm} F_i^m - \bar{H}_{ji}),$$

using (1.17) and (2.5) in equation (3.19), we have

$$(3.20) \quad S_{jm} F_i^m - S_{im} F_j^m = \bar{S}_{jm} F_i^m - \bar{S}_{im} F_j^m.$$

Thus we conclude

Theorem 3.2. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection if the Ricci tensor is pure with respect to Riemannian connection then it is also pure with respect quarter-symmetric metric connection if the equation (3.14) holds.*

In 2013, Arif Salimov and S. Turanli [6] defined

$$(3.21) \quad S_{ji}^* = -H_{jr} F_i^r = -R_{hjr l} G^{lh} F_i^r,$$

where S_{jr}^* is *Ricci tensor with respect to Riemannian connection.

Now we are taking

$$(3.22) \quad \bar{S}_{ji}^* = -\bar{H}_{jr} F_i^r = -\bar{R}_{hjr l} G^{lh} F_i^r,$$

where $\bar{S}_{jr}^*$ is *Ricci tensor with respect to quarter-symmetric metric connection.

With the help of equation (1.1), equation (3.21) and (3.22) can be written as

$$(3.23) \quad S_{jr}^* F_i^r = -H_{ji} \quad \text{and} \quad \bar{S}_{jr}^* F_i^r = -\bar{H}_{ji},$$

using equation (3.23) in (3.8), we have

$$(3.24) \quad (D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = (S_{jr} F_i^r + S_{jr}^* F_i^r) - (\bar{S}_{jm} F_i^m + \bar{S}_{jm}^* F_i^m),$$

from (3.24), if

$$(3.25) \quad D_h D_j F_i^h = \nabla_h \nabla_j F_i^h,$$

then, we have

$$(3.26) \quad S_{jr} F_i^r - \bar{S}_{jm} F_i^m = S_{jr}^* F_i^r - \bar{S}_{jm}^* F_i^m,$$

which implies $S_{jr} F_i^r = \bar{S}_{jm} F_i^m$, if only if $\bar{S}_{jm}^* F_i^m = S_{jr}^* F_i^r$.

Thus we conclude:

Theorem 3.3. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection the Ricci tensor with respect to Riemannian connection will be equal to Ricci tensor with respect to quarter-symmetric metric connection if only if *Ricci tensor with respect to Riemannian connection be equal to *Ricci tensor with respect to quarter symmetric metric connection if equations (3.14) and (3.25) hold.*

4. CONCLUSIONS

In this paper we have found that in a hyperbolic Kaehlerian manifold the Ricci tensor is pure with respect to quarter-symmetric metric connection if only if it is pure with respect to Riemannian connection with some conditions.

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