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ON RIESZ ALMOST LACUNARY CESÀRO $[C, 1, 1, 1]$ STATISTICAL CONVERGENCE IN PROBABILISTIC SPACE OF $\chi_f^{3\Delta}$

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Abstract. In this paper we study the concept of almost lacunary statistical Cesàro of χ^3 over probabilistic space P is defined by Musielak Orlicz function. Since the study of convergence in Probabilistic space P is fundamental to probabilistic functional analysis, we feel that the concept of almost lacunary statistical Cesàro of χ^3 over probabilistic space P is defined by Musielak in a probabilistic space P would provide a more general framework for the subject.

1. Introduction

A triple sequence (real or complex) can be defined as a function $x\colon \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}\,(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al.* [10, 11], *Esi et al.* [1, 2, 3], *Datta et al.* [4], *Subramanian et al.* [12], *Debnath et al.* [5] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The space of all triple entire sequences are usually denoted by Γ^3. The space Λ^3 and Γ^3 is a metric space with the metric

$$d(x, y) = \sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, \dots \right\}, \tag{1.1}$$

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for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\} in\Gamma^3$.

A complex sequence, whose k^{th} term x_k is denoted by $\{x_k\}$ or simply x. Let w be the set of all sequences $x = (x_k)$ and ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the sequence spaces of bounded,convergent and null sequences $x = (x_k)$ respectively. In respect of ℓ_∞, c, c_0 we have $\|x\| = \sup_k |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

The difference triple sequence space was introduced by Debnath et al. (see [5]) and is defined as

$\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$ and $\Delta^0 x_{mnk} = \langle x_{mnk} \rangle$.

2. Definitions and Preliminaries

Throughout the article $w^3, \chi^3(\Delta), \Lambda^3(\Delta)$ denote the spaces of all, triple gai difference sequence spaces and triple analytic difference sequence spaces respectively.

Subramanian et al. (see [12]) introduced by a triple entire sequence spaces, triple analytic sequences spaces and triple gai sequence spaces. The triple sequence spaces of $\chi^3(\Delta), \Lambda^3(\Delta)$ are defined as follows:

$\chi^3(\Delta) = \left\{x \in w^3 : ((m+n+k)! |\Delta x_{mnk}|)^{1/m+n+k} \to 0 \text{ as } m, n, k \to \infty\right\}$,

$\Lambda^3(\Delta) = \left\{x \in w^3 : \sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty\right\}$.

Definition 2.1. An Orlicz function (see [7]) is a function $M\colon [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([8]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup\{|v| u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [9]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \text{ as } m, n, k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(|x_{mnk}|\right)^{1/m+n+k}, \quad x=(x_{mnk})\in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an extended real number.

3. Further Definitions and Preliminaries

A triple sequence $x=(x_{mnk})$ has limit 0 (denoted by $P-limx=0$) (i.e) $P-\left((m+n+k)!\,|x_{mnk}|\right)^{1/m+n+k}\to 0$ as $m,n,k\to\infty$. We shall write more briefly as *P-convergent to 0.*

Definition 3.1. A triple sequence spaces of $x=(x_{mnk})$ of real numbers is called almost $P-$convergent to a limit 0 if

$$P-\lim_{p,q,u\to\infty}\sup_{r,s,t\geq 0}\frac{1}{pqu}\sum_{m=r}^{r+p-1}\sum_{n=s}^{s+q-1}\sum_{k=t}^{t+u-1}\left((m+n+k)!\,|x_{mnk}|\right)^{1/m+n+k}\to 0.$$

that is, the average value of (x_{mnk}) taken over any rectangle $\{(m,n,k): r\leq m\leq r+p-1, s\leq n\leq s+q-1, t\leq k\leq t+u-1\}$ tends to 0 as both p,q and u to ∞, and this $P-$convergence is uniform in i,ℓ and j. Let denote the set of sequences with this property as $\left[\widehat{\chi^3}\right]$.

Definition 3.2. Let $(q_m),(\overline{q_n}),(\overline{\overline{q_k}})$ be sequences of positive numbers and

$$Q_r=\begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0\ldots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \\ . & . & \ldots. & . & \ldots. \end{bmatrix}=q_{11}+q_{12}+\ldots+q_{rs}+\cdots\neq 0,$$

$$\overline{Q}_s=\begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0\ldots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \\ . & . & \ldots. & . & \ldots. \end{bmatrix}=\overline{q}_{11}+\overline{q}_{12}+\ldots+\overline{q}_{rs}+\cdots\neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0\ldots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0\ldots \\ . \\ . \\ . \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \cdots & \overline{\overline{q}}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \\ . & . & \ldots. & . & \ldots. \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} + \cdots \neq 0.$$

Then the transformation is given by

$$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left((m+n+k)! \left|x_{mnk}\right|\right)^{1/m+n+k}$$

is called the Riesz mean of triple sequence spaces of $x = (x_{mnk})$. If $P - \lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, then the triple sequence spaces of $x = (x_{mnk})$ is said to be Riesz convergent to 0. If the triple sequence spaces of $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - \lim x = 0$.

Definition 3.3. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty,$$

$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty,$$

$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determined by

$I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\}$, $q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.

Using the notations of lacunary sequence and Riesz mean for triple sequence spaces. $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m \in (0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k \in (0,k_j]} p_{k_j}$ and $H_i = \sum_{m \in (0,m_i]} p_{m_i}, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \left\{\left(Q_{m_i} Q_{n_j} Q_{k_k}\right)\right\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.

Throughout the paper, we assume that $Q_r = q_{11} + q_{12} + \ldots + q_{rs} + \cdots \to \infty\,(r \to \infty), \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} + \cdots \to \infty\,(s \to \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} +$

$\ldots + \overline{\overline{q}}_{rs} + \cdots \to \infty \, (t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.

Let $Q_{m_i, n_\ell, k_j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i \overline{H}_\ell \overline{\overline{H}}_j$,

$I'_{i\ell j} = \left\{ (m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \text{ and } \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j} \right\}$,

$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ and $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$, and $V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j$.

If we take $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.

4. Almost Lacunary Cesàro $[C, 1, 1, 1]$ −statistical convergence of probabilistic space P with triple sequence spaces of χ^3

Let $A = [a^{pqr}_{mnk}]^\infty_{m,n,k=0}$ be a triple infinite matrix of real number for $p, q, r = 1, 2, \ldots$ forming the sum

$$\mu_{pqr}(X) = \sum_{m=0}^\infty \sum_{n=0}^\infty \sum_{k=0}^\infty a^{pqr}_{mnk} \left(\left((m+n+k)! \left(\frac{x_{mnk}}{y_{mnk}} \right) \right)^{1/m+n+k}, \bar{0} \right) \tag{4.1}$$

called the A means of the triple sequence X yielded a method of summability. We say that a sequence X is A summable to the limit 0 of the A mean exist for all $p, q, r = 0, 1, \ldots$ and converges.

$$\lim_{uvw \to \infty} \sum_m^u \sum_n^v \sum_k^w a^{pqr}_{mnk} \left((m+n+k)! \left(\frac{x_{mnk}}{y_{mnk}} \right) \right)^{1/m+n+k} = \mu_{pqr}$$

and

$$\lim_{pqr \to \infty} \mu_{pqr} = 0$$

Define the means

$$\sigma^X_{pqr} = \frac{1}{pqr} \sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r \left((m+n+k)! \left(\frac{x_{mnk}}{y_{mnk}} \right) \right)^{1/m+n+k}$$

and

$$A\sigma^X_{pqr} = \frac{1}{pqr} \sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r a^{pqr}_{mnk} \left(\left((m+n+k)! \left(\frac{x_{mnk}}{y_{mnk}} \right) \right)^{1/m+n+k}, \bar{0} \right).$$

We say that $\left(\frac{x_{mnk}}{y_{mnk}} \right)$ is statistically lacunary equivalent summable $[C, 1, 1, 1]$ to 0, if the sequence $\sigma = (\sigma^x_{mnk})$ is statistically convergent to $\bar{0}$, that is, $st_3 - \lim_{pqr} \sigma^x_{pqr} = 0$. It is denoted by $[C, 1, 1, 1]\,(st_3)$, the set of all triple sequence which one statistically lacunary equivalent to summable $[C, 1, 1, 1]$.

Let $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$. If we choose $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k.

Definition 4.1. A triple $(X, P, *)$ be a probabilistic space. Then a triple sequence spaces $x = (x_{mnk})$ is said to statistically convergent to $\bar{0}$ with respect to the probabilistic, $P-$ provided that for every $\epsilon > 0$ and $\gamma \in (0, 1)$

$$\delta\left(\left\{m, n, k \in \mathbb{N} : P - \lim_{r,s,t\to\infty} \frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon)\right] \leq \right.\right.$$

$$\left.\left. \leq 1 - \gamma \right\}\right) = 0$$

or equivalently $\lim_{k\ell v} \frac{1}{k\ell v} m \leq k, n \leq \ell, k \leq v :$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon)\right] \leq 1 - \gamma = 0$$

In this case we write $St_P - \lim_x = \bar{0}$.

Definition 4.2. A triple sequence spaces of $(X, P, *)$ be a probabilistic space P. The two non-negative triple sequence spaces of $x = (x_{mnk})$ and $y = (y_{mnk})$ are said to be almost asymptotically statistical equivalent of multiple $\bar{0}$ in probabilistic space X if for every $\epsilon > 0$ and $\gamma \in (0, 1)$.

$$\delta\left(\left\{ m, n \in \mathbb{N} : \right.\right.$$

$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] \leq 1 - \gamma\right\}\right) = 0$$

or equivalently

$$\lim_{k\ell} \frac{1}{k\ell} \left|\left\{ m \leq k, n \leq \ell : \right.\right.$$

$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] \leq 1 - \gamma\right\}\right| = 0.$$

In this case we write $x \overset{\widehat{S}(P)}{\equiv} y$.

Definition 4.3. A triple sequence spaces of $(X, P, *)$ be a probabilistic space P and $\theta = (m_r n_s k_t)$ be a triple lacunary sequence spaces are said to be a almost asymptotically lacunary statistical equivalent of multiple $\bar{0}$ in probabilistic space X if for every $\epsilon > 0$ and $\gamma \in (0, 1)$

$$\delta_\theta\left(\left\{ m, n, k \in I_{rst} : \right.\right. \tag{4.2}$$

$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] \leq 1 - \gamma\right\}\right) = 0$$

or equivalently

$$\lim_{rst} \frac{1}{h_{rst}} \Bigg| \Bigg\{ \; m,n \in I_{rst} :$$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr} \right)(\epsilon), \bar{0} \right] \le 1 - \gamma \Bigg\} \Bigg| = 0.$$

In this case we write $x \overset{\widehat{S_\theta(P)}}{\equiv} y$.

Lemma 4.4. *A triple sequence spaces of $(X, P, *)$ be a probabilistic space P. Then for every $\epsilon > 0$ and $\gamma \in (0,1)$, the following statements are equivalent:*

(1) $$\lim_{rst} \frac{1}{h_{rst}} \Bigg| \Bigg\{ \; m,n,k \in I_{rst} :$$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr} \right)(\epsilon), \bar{0} \right] \le 1 - \gamma \Bigg\} \Bigg| = 0,$$

(2) $$\delta_\theta \Bigg(\Bigg\{ \; m,n,k \in I_{rst} :$$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr} \right)(\epsilon), \bar{0} \right] \le 1 - \gamma \Bigg\} \Bigg) = 0,$$

(3) $$\delta_\theta \Bigg(\Bigg\{ \; m,n,k \in I_{rst} :$$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr} \right)(\epsilon), \bar{0} \right] \le 1 - \gamma \Bigg\} \Bigg) = 1,$$

(4) $$\lim_{rst} \frac{1}{h_{rst}} \Bigg| \Bigg\{ \; m,n,k \in I_{rst} :$$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr} \right)(\epsilon), \bar{0} \right] \le 1 - \gamma \Bigg\} \Bigg| = 1.$$

5. Main Results

Theorem 5.1. *A triple sequence spaces of $(X, P, *)$ be a probabilistic space P. If two non-negative triple sequence spaces of $x = (x_{mnk})$ and $y = (y_{mnk})$ are almost asympototically lacunary statistical equivalent of multiple $\bar{0}$ with respect to the probabilistic P, then $\bar{0}$ is unique sequence.*

Proof. Assume that $x \overset{\widehat{S^{\bar{0}}_{\theta}}(P)}{\equiv} y$. For a given $\lambda > 0$ choose $\gamma \in (0,1)$ such that $(1-\gamma) > 1-\lambda$. Then, for any $\epsilon > 0$, define the following set:

$$K = \Bigg\{ \; m,n \in I_{r,s} :$$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr}\right)(\epsilon), \bar{0}\right] \leq 1-\gamma \Bigg\}$$

Then, clearly

$$\lim_{rst} \frac{K \bigcap \bar{0}}{h_{rst}} = 1,$$

so K is non-empty set, since $x \overset{\widehat{S^{\bar{0}}_{\theta}}(P)}{\equiv} y, \delta_\theta(K) = 0$ for all $\epsilon > 0$, which implies $\delta_\theta(\mathbb{N} - K) = 1$. If $m,n,k \in \mathbb{N} - K$, then we have

$$\begin{aligned} P_{\bar{0}}(\epsilon) &= P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr}\right)(\epsilon), \bar{0}\right] \\ &> (1-\gamma) \geq 1-\lambda \end{aligned}$$

since λ is arbitrary, we get $P_{\bar{0}}(\epsilon) = 1$.

This completes the proof. □

Theorem 5.2. *A triple sequence spaces of* $(X, P, *)$ *be a probabilistic space. For any lacunary sequence* $\theta = (m_r n_s k_t)$, $\widehat{S}_\theta(P) \subset \widehat{S}(P)$ *if* $\limsup_{rst} q_{rst} < \infty$.

Proof. If $\limsup\limits_{rst} q_{rst} < \infty$. then there exists a $B > 0$ such that $q_{rst} < B$ for all $r,s,t \geq 1$. Let $x \overset{\widehat{S_{\theta}}(P)}{\equiv} y$ and $\epsilon > 0$. Now we have to prove $\widehat{S}(P)$. Set

$$K_{rst} = \Bigg|\Bigg\{ \; m,n,k \in I_{rst} :$$

$$P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma^x_{pqr}\right)(\epsilon), \bar{0}\right] 1-\gamma \Bigg\}\Bigg|.$$

Then by definition, for given $\epsilon > 0$, there exists $r_0 s_0 t_0 \in \mathbb{N}$ such that

$$\frac{K_{rst}}{h_{rst}} < \frac{\epsilon}{2B} \text{ for all } r > r_0,\; s > s_0 \text{ and } t > t_0.$$

Let

$$M = \max\{K_{rs} :\; 1 \leq r \leq r_0,\; 1 \leq s \leq s_0,\; 1 \leq t \leq t_0\}$$

and let uvw be any positive integer with $m_{r-1} < u \leq m_r, n_{s-1} < v \leq n_s$ and $k_{t-1} < w \leq k_t$. Then

$$\frac{1}{uvw}\left|\left\{ m \leq u, n \leq v, k \leq w : \right.\right.$$
$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] > 1-\gamma \right\}\right|$$
$$\leq \frac{1}{m_{r-1}n_{s-1}k_{t-1}}\left|\left\{ m \leq m_r, n \leq n_s, k \leq k_t : \right.\right.$$
$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] > 1-\gamma \right\}\right|$$
$$= \frac{1}{m_{r-1}n_{s-1}k_{t-1}} \left\{K_{111} + \ldots + K_{rst}\right\}$$
$$\leq \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\epsilon}{2B} q_{rst} \leq \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\epsilon}{2}.$$

This completes the proof. □

Theorem 5.3. *A triple sequence spaces of* $(X, P, *)$ *be a probabilistic space* P*. For any lacunary sequence* $\theta = (m_r n_s k_t)$, $\widehat{S}(P) \subset \widehat{S}_\theta(P)$ *if* $\liminf_{rst} q_{rst} > 1$.

Proof. If $\liminf_{rst} q_{rst} > 1$, then there exists a $\beta > 0$ such that $q_{rst} > 1 + \beta$ for sufficiently large rst, which implies

$$\frac{h_{rst}}{K_{rst}} \geq \frac{\beta}{1+\beta}.$$

Let $x \overset{\widehat{S^{\bar{0}}}(p)}{\equiv} y$, then for every $\epsilon > 0$ and for sufficiently large r, s, t we have

$$\frac{1}{m_r n_s k_t}\left|\left\{ m \leq m_r, n \leq n_s, k \leq k_t : \right.\right.$$
$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] > 1-\gamma \right\}\right|$$
$$\geq \frac{1}{m_r n_s k_t}\left|\left\{ m, n, k \in I_{rst} : \right.\right.$$
$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] > 1-\gamma \right\}\right|$$
$$\geq \frac{\beta}{1+\beta} \frac{1}{h_{rst}}\left|\left\{ m, n, k \in I_{rst} : \right.\right.$$
$$\left.\left. P - \lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^{x}\right)(\epsilon), \bar{0}\right] > 1-\gamma \right\}\right|.$$

Therefore $x \overset{\widehat{S^{0}_{\theta}}(p)}{\equiv} y$. This completes the proof. □

Corollary 5.4. *A triple sequence spaces of* $(X, P, *)$ *be a probabilistic space* P *. For any lacunary sequence* $\theta = (m_r n_s k_t)$ *, with* $1 < \liminf_{rst} q_{rst} \leq \limsup_{rst} q_{rst} < \infty$*, then* $\widehat{S}(P) = \widehat{S}_{\theta}(P)$ *.*

Proof. The result clearly follows from Theorem 4.2 and Theorem 4.3. □

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