

LEGENDRE CURVES ON THREE-DIMENSIONAL QUASI-SASAKIAN MANIFOLDS WITH SEMI SYMMETRIC METRIC CONNECTION

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ABSTRACT. The object of the present paper is to study Legendre curves on three-dimensional quasi-Sasakian manifolds with semi-symmetric metric connection.

1. INTRODUCTION

In the study of contact manifolds, Legendre curve play an important role e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair [1]. Legendre curves with Pseudo-Hermitian connection have been studied by J. T. Cho [5]. The first author of the paper has studied Legendre curves in the papers [11], [12]. Again Legendre curves on three-dimensional quasi-Sasakian manifold has been studied in the paper [2]. In this paper we are interested to study Legendre curves on three-dimensional quasi-Sasakian manifolds with respect to semi-symmetric metric connection. The notion of quasi-Sasakian manifolds was given by D. E. Blair in the paper [4]. Again Z. Olszak [10] studied quasi-Sasakian manifolds. Semi-symmetric metric connection was studied by K. Yano [14]. Semi-symmetric connection was introduced by Schouten [7]. Later Hayden [8] has introduced the idea of metric connection with torsion in a Riemannian manifold. A. Sharfuddin and S. I. Hussain [13] introduced the study of almost contact manifolds with semi-symmetric metric connection. The present paper is organized as follows:

After the introduction we give some preliminaries in Section 2. Section 3 is devoted to study biharmonic Legendre curves on three-dimensional quasi-Sasakian manifolds with respect to semi symmetric metric connection. In

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Section 4, we study locally ϕ -symmetric Legendre curves on three-dimensional quasi-Sasakian manifolds with respect to semi-symmetric metric connection.

2. PRELIMINARIES

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) i.e., ϕ is a 1-1 tensor field, ξ is a unit vector field, η is a 1-form and g is a Riemannian metric such that [3]

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi) = 0$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in \chi(M)$.

An almost contact metric manifold of dimension three is quasi-Sasakian if and only if

$$(2.3) \quad \nabla_X \xi = -\beta \phi X,$$

for $X \in \chi(M)$ and a function β defined on the manifold [10].

As a consequence of (2.3), we have [9]

$$(2.4) \quad (\nabla_X \phi)Y = \beta(g(X, Y)\xi - \eta(Y)X), \quad X, Y \in \chi(M)$$

$$(2.5) \quad (\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y)$$

$$(2.6) \quad (\nabla_X \eta)\xi = -\beta \eta(\phi X) = 0$$

The curvature tensor of a three-dimensional quasi-Sasakian manifold is given by [6]

$$(2.7) \quad \begin{aligned} R(X, Y)Z = & g(Y, Z)[(\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi + \eta(X)(\phi \text{ grad } \beta) \\ & - d\beta(\phi X)\xi] - g(X, Z)[(\frac{r}{2} - \beta^2)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi \\ & + \eta(Y)(\phi \text{ grad } \beta) - d\beta(\phi Y)\xi] + [(\frac{r}{2} - \beta^2)g(Y, Z) \\ & + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y)]X \\ & - [(\frac{r}{2} - \beta^2)g(X, Z) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Z) - \eta(X)d\beta(\phi Z) \\ & - \eta(Z)d\beta(\phi X)]Y - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

A curve γ on a manifold M is called Legendre curve if it satisfies [1]

$$(2.8) \quad \eta(\dot{\gamma}) = 0$$

The semi symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on an almost contact metric manifold are related by

$$(2.9) \quad \tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi$$

for all vector fields X, Y on M .

The torsion tensor of a semi symmetric metric connection on an almost contact metric manifold is given by

$$(2.10) \quad \tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y$$

A curve γ on M is called Frenet curve with respect to semi-symmetric metric connections if it satisfies

$$(2.11) \quad \tilde{\nabla}_T T = \tilde{k}N$$

$$(2.12) \quad \tilde{\nabla}_T N = -\tilde{k}T + \tilde{\tau}B$$

$$(2.13) \quad \tilde{\nabla}_T B = -\tilde{\tau}N$$

where $\tilde{k}, \tilde{\tau}$ are the curvature and torsion of the curve with respect to semi symmetric metric connection, $\{T, N, B\}$ is an orthonormal frame with $\dot{\gamma} = T$.

3. BIHARMONIC LEGENDRE CURVES WITH RESPECT TO SEMI SYMMETRIC METRIC CONNECTION

Definition 3.1. A Legendre curve on three-dimensional quasi-Sasakian manifold will be called biharmonic with respect to semi-symmetric metric connection if it satisfies [5]

$$(3.1) \quad \tilde{\nabla}_T^3 T + \tilde{\nabla}_T \tilde{\tau}(\tilde{\nabla}_T T, T)T + \tilde{R}(\tilde{\nabla}_T T, T)T = 0$$

where $\tilde{\tau}$ is torsion of semi symmetric connection and T is tangent vector field of the curve.

Let us consider a Legendre curve γ and T be the tangent. We take $T, \phi T, \xi$ as the orthonormal right handed system where $\phi T = -N, \phi N = T$. For semi-symmetric metric connection, we have $\tilde{\nabla}_T \tilde{\tau}(\tilde{\nabla}_T T, T)T = 0$.

Hence (3.1) reduces to

$$(3.2) \quad \tilde{\nabla}_T^3 T + \tilde{k}\tilde{R}(N, T)T = 0.$$

Let $\tilde{R}$ and R be the curvature tensor of a three-dimensional quasi-Sasakian manifold with respect to semi-symmetric metric connection and Levi-Civita connection respectively. Then the relation between $\tilde{R}$ and R is given by [14]

$$(3.3) \quad \begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z - L(Y, Z)X + L(X, Z)Y \\ &+ 2g(\nabla_Y X, Z)\xi - 2g(\nabla_X Y, Z)\xi + \eta(Z)([X, Y]) \\ &+ \eta(X)g(Y, Z)\xi + \eta(Y)g(X, Z)\xi, \end{aligned}$$

where

$$(3.4) \quad L(Y, Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + g(Y, Z)\xi$$

Now using (2.5) in (3.4) we get

$$(3.5) \quad L(Y, Z) = -\beta g(\phi Y, Z) - \eta(Y)\eta(Z) + g(Y, Z)\xi$$

Using (3.5) in (3.3) we get

$$\begin{aligned}
 (3.6) \quad \tilde{R}(X, Y)Z &= R(X, Y)Z + \beta(g(\phi Y, Z)X - g(\phi X, Z)Y) \\
 &\quad + \eta(Z)(\eta(Y)X - \eta(X)Y) - (g(Y, Z)X - g(X, Z)Y) \\
 &\quad + 2(g(\nabla_Y X, Z) - g(\nabla_X Y, Z))\xi + (\eta(X)g(Y, Z)\xi + \eta(Y)g(X, Z)\xi) \\
 &\quad + \eta(Z)([X, Y])
 \end{aligned}$$

Since we have considered Frenet Frame as $T, \phi T, \xi$ where $\phi T = -N$, so for a Legendre curve we get $\eta(T) = 0$, $\eta(N) = 0$. Using this fact and putting $X = N$, $Y = T$, $Z = T$ in (3.6) we get

$$(3.7) \quad \tilde{R}(N, T)T = R(N, T)T - N + \beta T + 2[g(\nabla_T N, T) - g(\nabla_N T, T)]\xi$$

Now putting $X = N$, $Y = T$, $Z = T$ in (2.7) we get

$$(3.8) \quad R(N, T)T = \frac{r}{2}N - 2\beta^2 N - d\beta(\phi N)\xi$$

From (3.7) and (3.8) after some simplification and setting $\xi = B$ we get

$$(3.9) \quad \tilde{R}(N, T)T = \frac{r}{2}N - 2\beta^2 N - d\beta(\phi N)B - N + \beta T - 2\tilde{k}B$$

Again by Serret-Frenet formula we get,

$$(3.10) \quad \tilde{\nabla}^3_T T = -3\tilde{k}\tilde{k}'T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2)N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}')B$$

From (3.9) and (3.10) we get,

$$\begin{aligned}
 \tilde{\nabla}^3_T T + \tilde{k}\tilde{R}(N, T)T &= (-3\tilde{k}\tilde{k}' - \tilde{k}\beta)T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 + \tilde{k}\frac{r}{2} - 2\tilde{k}\beta^2 + \tilde{k})N \\
 &\quad + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' - \tilde{k}d\beta(\phi N) + 2\tilde{k}^2)B.
 \end{aligned}$$

If the Legendre curve is biharmonic, then we have $\tilde{\nabla}^3_T T + \tilde{k}\tilde{R}(N, T)T = 0$. So we have

$$(3.11) \quad -3\tilde{k}\tilde{k}' - \tilde{k}\beta = 0$$

$$(3.12) \quad \tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 + \tilde{k}\frac{r}{2} - 2\tilde{k}\beta^2 + \tilde{k} = 0$$

$$(3.13) \quad 2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' - \tilde{k}d\beta(\phi N) + 2\tilde{k}^2 = 0.$$

In view of (3.11), we obtain the following theorem:

Theorem 3.1. *The curvature of a non-geodesic biharmonic Legendre curve on a three-dimensional quasi-Sasakian manifold with respect to semi-symmetric connection is given by $\tilde{k} = -\frac{1}{3} \int \beta ds$, where s is the arc length parameter.*

4. LOCALLY ϕ -SYMMETRIC LEGENDRE CURVES

Definition 4.1. With respect to semi-symmetric metric connection a Legendre curve on a three-dimensional quasi-Sasakian manifold is called locally ϕ -symmetric if it satisfies [11]

$$(4.1) \quad \phi^2(\tilde{\nabla}_T \tilde{R})(\tilde{\nabla}_T T, T)T = 0$$

Now putting $X = \tilde{\nabla}_T T$, $Y = Z = T$ in (3.6) and (2.7) and then using Serret-Frenet formula, after some calculations we get

$$(4.2) \quad \tilde{R}(\tilde{\nabla}_T T, T)T = \beta \tilde{k}T + \left(\frac{r}{2}\tilde{k} - 2\beta\tilde{k} - \tilde{k}\right)N - (\tilde{k}d\beta(\phi N) + 2\tilde{k}^2)B.$$

Again putting $X = B$, $Y = Z = T$ in (3.6) and (2.7) and then using $\phi T = -N$ we get

$$(4.3) \quad \tilde{R}(B, T)T = \beta^2 B + \phi \operatorname{grad} \beta + d\beta(N)T.$$

By definition of covariant differentiation of $\tilde{R}$ and using Serret-Frenet formula, we get

$$(4.4) \quad (\tilde{\nabla}_T \tilde{R})(\tilde{\nabla}_T T, T)T = \tilde{\nabla}_T \tilde{R}(\tilde{\nabla}_T T, T)T \\ - \tilde{k}\tilde{\tau}\tilde{R}(B, T)T - \tilde{k}\tilde{R}(N, T)T - \tilde{k}^2\tilde{R}(N, T)N$$

Again putting $X = N$, $Y = T$, $Z = N$ in (3.6) and (2.7) and setting $\xi = B$ we get

$$(4.5) \quad \tilde{R}(N, T)N = 2\beta^2 T - \frac{r}{2}T + d\beta(N)B - \beta N + T - 2g(\nabla_N T, N)B$$

Now using (4.2) and Serret-Frenet formula we get

$$(4.6) \quad \tilde{\nabla}_T \tilde{R}(\tilde{\nabla}_T T, T)T = [(\beta\tilde{k})' - \frac{r}{2}\tilde{k}^2 + 2\beta\tilde{k}^2 + \tilde{k}^2]T \\ + [\beta\tilde{k}^2 + (\frac{r}{2}\tilde{k})' - 2(\beta\tilde{k})' - \tilde{k}' + \tilde{k}d\beta(\phi N)\tilde{\tau} \\ + 2\tilde{k}^2\tilde{\tau}]N + [\frac{r}{2}\tilde{k}\tilde{\tau} - 2\beta\tilde{k}\tilde{\tau} - \tilde{k}\tilde{\tau} \\ - \tilde{k}\tilde{\nabla}_T(d\beta(\phi N))\tilde{k}'d\beta(\phi N) - 4\tilde{k}\tilde{k}']B$$

Now from (3.9), (4.3), (4.4), (4.5), (4.6) we get

$$(4.7) \quad (\tilde{\nabla}_T \tilde{R})(\tilde{\nabla}_T T, T)T = [(\beta\tilde{k})' + 2\beta\tilde{k}^2 - \tilde{k}\tilde{\tau}d\beta(N) - \tilde{k}\beta - 2\beta^2\tilde{k}]T \\ + [\beta\tilde{k}^2 + (\frac{r}{2}\tilde{k})' - 2(\beta\tilde{k})' - \tilde{k}' + \tilde{k}d\beta(\phi N)\tilde{\tau} \\ + 2\tilde{k}^2\tilde{\tau} - \tilde{k}\frac{r}{2} + 2\tilde{k}\beta^2 + \tilde{k}^2\beta]N \\ + [\frac{r}{2}\tilde{k}\tilde{\tau} - 2\beta\tilde{k}\tilde{\tau} - \tilde{k}\tilde{\tau} - \tilde{k}\tilde{\nabla}_T(d\beta(\phi N)) \\ - \tilde{k}'d\beta(\phi N) - 4\tilde{k}\tilde{k}' - \tilde{k}\tilde{\tau}\beta^2 + \tilde{k}d\beta(\phi N) + 2\tilde{k}^2 \\ - \tilde{k}^2d\beta(N) + 2\tilde{k}^2g(\nabla_N T, N)]B - \tilde{k}\tilde{\tau}\phi \operatorname{grad} \beta$$

Applying ϕ^2 on both sides, we get,

$$\begin{aligned}
 \phi^2(\tilde{\nabla}_T \tilde{R})(\tilde{\nabla}_T T, T)T = & -[(\beta \tilde{k})' + 2\beta \tilde{k}^2 - \tilde{k} \tilde{\tau} d\beta(N) - \tilde{k} \beta - 2\beta^2 \tilde{k}]T \\
 & - [\beta \tilde{k}^2 + (\frac{r}{2} \tilde{k})' - 2(\beta \tilde{k})' - \tilde{k}' + \tilde{k} d\beta(\phi N) \tilde{\tau} \\
 (4.8) \quad & + 2\tilde{k}^2 \tilde{\tau} - \tilde{k} \frac{r}{2} + 2\tilde{k} \beta^2 + \tilde{k}^2 \beta]N \\
 & - \tilde{k} \tilde{\tau} \phi^3 \text{grad } \beta
 \end{aligned}$$

If the curves are locally ϕ -symmetric, then $\tilde{k} \tilde{\tau} \phi^3 \text{grad } \beta = 0$.

Let $\tilde{k} \neq 0$ and β is not constant. Then $\tilde{\tau} = 0$. So, the torsion with respect to semi-symmetric connection of a locally ϕ -symmetric Legendre curve on a three-dimensional quasi-Sasakian manifold is zero.

Theorem 4.1. *A non-geodesic locally ϕ -symmetric Legendre curve with respect to semi-symmetric metric connection on a three-dimensional quasi-Sasakian manifold with non-constant structure function is a plane curve.*

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