

ON $N(k)$ -MIXED-SUPER QUASI-EINSTEIN MANIFOLDS SATISFYING SOME CONDITIONS

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ABSTRACT. In this paper $N(k)$ -mixed super quasi Einstein manifold $N(k) - MS(QE)_n$ has been introduced and the existence of such manifold is proved. Here, we have studied the nature of Ricci curvature, Ricci symmetric, Ricci recurrent, Generalized Ricci recurrent $N(k) - MS(QE)_n$. Next we study when the curvature conditions $\tilde{C}(U, X).S = 0$ and $\tilde{P}(U, X).S = 0$ hold in $N(k) - MS(QE)_n$ where $\tilde{C}$ and $\tilde{P}$ are the concircular curvature tensor and Weyl projective curvature tensor. We also study the Ricci-pseudosymmetric $N(k) - MS(QE)_n$. Finally, we give an example of $N(k) - MS(QE)_n$.

1. INTRODUCTION

The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [3]. A non-flat Riemannian manifold (M^n, g) , $(n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$(1.1) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y)$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$(1.2) \quad g(X, U) = A(X), \quad \forall X \in \chi(M),$$

where $\chi(M)$ is the set of all differentiable vector fields on M and U being a unit vector field.

Here a and b are called the *associated scalars*, A is called the *associated 1-form* and U is called the *generator* of the manifold. Such an n -dimensional manifold will be denoted by $(QE)_n$.

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In [4], Authors have defined *generalized quasi-Einstein manifold*. A non-flat *Riemannian manifold* is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$(1.3) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b and c are non-zero scalars and A, B are two 1-forms such that

$$(1.4) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X),$$

U and V being unit vectors which are orthogonal, i.e.,

$$(1.5) \quad g(U, V) = 0.$$

The vector fields U and V are called the *generators* of the manifold. This type of manifold will be denoted by $G(QE)_n$.

In [2], Chaki introduced the *super quasi-Einstein manifold*, denoted by $S(QE)_n$, where the Ricci tensor is not identically zero and satisfies the condition

$$(1.6) \quad \begin{aligned} S(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) \\ + A(Y)B(X)] + dD(X, Y), \end{aligned}$$

where a, b, c and d are scalars such that b, c, d are nonzero, A, B are two nonzero 1-forms defined as (1.4) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$(1.7) \quad D(X, U) = 0 \quad \forall X \in \chi(M).$$

Here a, b, c, d are called the *associated scalars*, A, B are called the *associated main and auxiliary 1-forms* respectively, U, V are called the *main and the auxiliary generators* and D is called the *associated tensor* of the manifold.

The k -nullity distribution $N(k)$ [11] of a *Riemannian manifold* M is defined by

$$N(k) : p \rightarrow N_p(k) = \{Z \in T_p M / R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)\}$$

for all $X, Y \in \chi(M)$ and k is a smooth function.

M. M. Tripathi and Jeong-Sik Kim [12] introduced the notion of $N(k)$ -*quasi Einstein manifold* which is defined as follows: If the generator U belongs to the k -nullity distribution $N(k)$, then a *quasi Einstein manifold* (M^n, g) is called $N(k)$ -*quasi Einstein manifold*. In [9], Nagaraja introduced the notion of $N(k)$ -*mixed quasi Einstein manifold*.

In [1], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of $MS(QE)_n$. So, we define $N(k) - MS(QE)_n$ as follows:

Definition. Let (M^n, g) be a non flat Riemannian manifold. If the Ricci tensor S of (M^n, g) is non zero and satisfies

$$(1.8) \quad \begin{aligned} S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) \\ + A(Y)B(X)] + eD(X, Y), \end{aligned}$$

where a, b, c, d, e are scalars of which $b \neq 0, c \neq 0, d \neq 0, e \neq 0$ and A, B are two non zero 1-forms such that

$$(1.9) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X) \quad \forall X \in \chi(M),$$

D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$(1.10) \quad D(X, U) = 0 \quad \forall X \in \chi(M),$$

U and V being the orthogonal unit vector fields called *generators* of the manifold belong to $N(k)$, then we say that (M^n, g) is a $N(k)$ -mixed super quasi Einstein manifold and is denoted by $N(k) - MS(QE)_n$.

2. PRELIMINARIES

In $N(k) - MS(QE)_n$, we have

$$(2.1) \quad R(X, Y)U = k\{A(Y)X - A(X)Y\}.$$

From (1.8), we have

$$(2.2) \quad S(U, U) = a + b$$

$$(2.3) \quad S(V, V) = a + c + eD(V, V)$$

$$(2.4) \quad S(U, V) = d = S(V, U).$$

Now setting $X = Y = e_i$ in (1.8), where $\{e_i\}, i = 1, 2, \dots, n$ be an orthonormal basis of vector fields in the manifold and taking summation over $i, 1 \leq i \leq n$, we obtain

$$(2.5) \quad r = na + b + c,$$

where r is the scalar curvature of the manifold.

If X is a unit vector field, then $S(X, X)$ is the Ricci-curvature in the direction of X . Hence from (2.2) and (2.3) we can state that $a + b$ and $a + c + eD(V, V)$ are the Ricci curvature in the directions of U and V respectively.

Let Q be the Ricci operator, i.e.,

$$(2.6) \quad g(QX, Y) = S(X, Y) \quad \forall X, Y \in \chi(M).$$

Here, we consider

$$(2.7) \quad g(lX, Y) = D(X, Y).$$

3. EXISTENCE THEOREM OF A $N(k)$ -MIXED SUPER QUASI EINSTEIN MANIFOLD $N(k) - MS(QE)_n$

Theorem 3.1. *If in a conformally flat Riemannian manifold (M^n, g) , the Ricci tensor S satisfies the relation*

$$(3.1) \quad \begin{aligned} S(X, W)S(Y, Z) - S(Y, W)S(X, Z) = & \mu_1[S(Y, W)g(Z, X) + S(Z, X) \\ & g(Y, W)] + \beta_1[g(X, W)g(Y, Z) - g(Y, W)g(Z, X)] \\ & + \gamma_1[g(Y, Z)D(X, W) - g(X, Z)D(Y, W) \\ & + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)], \end{aligned}$$

where μ_1, β_1, γ_1 are non-zero scalars and D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition $D(X, U) = 0, \forall X$ then the manifold is $N(k)$ -mixed super quasi-Einstein manifold.

Proof. Existence theorem of a mixed super quasi Einstein manifold was proved in [1]. Now, we will prove the Existence Theorem of a $N(k)$ -mixed super quasi Einstein manifold.

If (M^n, g) is conformally flat, then

$$(3.2) \quad \begin{aligned} R(X, Y)Z = & \frac{1}{n-1}\{g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y\} \\ & - \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned}$$

Taking $Z = U$ in (3.2), we obtain

$$(3.3) \quad \begin{aligned} R(X, Y)U = & \frac{1}{n-1}\{A(Y)QX - A(X)QY + S(Y, U)X - S(X, U)Y\} \\ & - \frac{r}{(n-1)(n-2)}\{A(Y)X - A(X)Y\}. \end{aligned}$$

Now taking $\mu_1 = \beta_1 = \gamma_1$ and $Z = U$ in (3.1), we obtain

$$(3.4) \quad \begin{aligned} S(X, W)[(a+b)A(Y) + dB(Y)] - S(Y, W)[(a+b)A(X) + dB(X)] \\ = S(Y, W)A(X) + [(a+b)A(X) + dB(X)]g(Y, W) \\ + g(X, W)A(Y) - g(Y, W)A(X) + D(X, W)A(Y) \\ - D(Y, W)A(X). \end{aligned}$$

Now taking $a+b=1$ and $d=1$ and using $S(X, W) = g(QX, W)$ and $D(X, W) = g(lX, W)$ in (3.4), we get

$$(3.5) \quad \begin{aligned} g((QX)A(Y) + (QX)B(Y) - (QY)A(X) - (QY)B(X) - (QY)A(X) \\ - A(X)Y - B(X)Y - A(Y)X + A(X)Y \\ - A(Y)lX + A(X)lY, W) = 0. \end{aligned}$$

$\forall W$, which implies

$$(3.6) \quad \begin{aligned} (QX)A(Y) + (QX)B(Y) - (QY)A(X) - (QY)B(X) - (QY)A(X) \\ - A(X)Y - B(X)Y - A(Y)X + A(X)Y \\ - A(Y)lX + A(X)lY = 0. \end{aligned}$$

Or,

$$(3.7) \quad \begin{aligned} (QX)A(Y) - (QY)A(X) + [A(Y) + B(Y)]X - [A(X) + B(X)]Y \\ = A(Y)X - A(X)Y, \end{aligned}$$

where

$$(3.8) \quad \begin{aligned} [A(Y) + B(Y)]X - [A(X) + B(X)]Y = [-A(Y)lX + B(Y)QX] \\ - [A(X)\{QY + Y - lY\} + B(X)(QY + Y)]. \end{aligned}$$

Substituting (3.7) in (3.3), we get

$$(3.9) \quad R(X, Y)U = k[A(Y)X - A(X)Y],$$

where, $k = \frac{n(1-a)-b-c-1}{(n-1)(n-2)}$.

Therefore, $U \in N_p(k)$ for $k = \frac{n(1-a)-b-c-1}{(n-1)(n-2)}$.

Hence (M^n, g) is a $N(k)$ -mixed super quasi Einstein manifold. $\square$

As it is well known that a 3-dimensional Riemannian manifold is conformally flat.

Corollary. *A 3-dimensional manifold is $N(\frac{2-3a-b-c}{2})$ mixed super quasi Einstein manifold provided (3.1) holds.*

4. RICCI CURVATURE, EIGEN VECTORS AND ASSOCIATED SCALARS OF A $N(k) - MS(QE)_n$

From (1.8), we have $S(U, U) = a+b$, $S(V, V) = a+c+eD(V, V)$, $S(U, V) = d$, $S(X, X)$ is the Ricci curvature in the direction of X . Now,

$$(4.1) \quad 1 = g(X, X) = g(\alpha U + \beta V, \alpha U + \beta V) = \alpha^2 + \beta^2.$$

Since $g(U, V) = 0$ and $g(U, U) = g(V, V) = 1$.

Now,

$$(4.2) \quad S(X, X) = a + b\{A(X)\}^2 + c\{B(X)\}^2 + 2dA(X)B(X) + eD(X, X).$$

Thus, we can state the following theorem:

Theorem 4.1. *In a $N(k) - MS(QE)_n$ manifold, the Ricci curvature in the direction of U is $a+b$ and in the direction of V is $a+c+eD(V, V)$ and the Ricci curvature in all other directions of the section of U and V is $a + b\{A(X)\}^2 + c\{B(X)\}^2 + 2dA(X)B(X) + eD(X, X)$.*

Let (M^n, g) be $N(k) - MS(QE)_n$, then we get

$$\begin{aligned} S(U, U) &= a + b, \quad S(V, V) = a + c + eD(V, V), \quad S(U, V) = d \\ g(QU, U) &= a + b, \quad g(QV, V) = a + c + eD(V, V). \end{aligned}$$

Since $U, V \in N_p(k)$, we have

$$(4.3) \quad g(R(X, Y)U, W) = k\{g(Y, U)g(X, W) - g(X, U)g(Y, W)\}.$$

From (1.9),

$$(4.4) \quad g(R(X, Y)U, W) = k\{A(Y)g(X, W) - A(X)g(Y, W)\}.$$

Putting $X = W = e_i$ in (4.4) where $\{e_i\}$, $i = 1, 2, \dots, n$ be an orthonormal basis of the tangent space at any point of the manifold and taking summation over i , $1 \leq i \leq n$, we obtain

$$(4.5) \quad S(Y, U) = k(n-1)A(Y),$$

and

$$(4.6) \quad S(Y, V) = k(n-1)B(Y).$$

Again from (1.8), we get

$$(4.7) \quad S(Y, U) = (a+b)A(Y) + dB(Y).$$

$$(4.8) \quad S(Y, V) = (a+c)B(Y) + dA(Y) + eD(Y, V).$$

Subtracting (4.6) from (4.5), we obtain

$$(4.9) \quad S(Y, U) - S(Y, V) = k(n-1)[A(Y) - B(Y)].$$

Subtracting (4.8) from (4.7), we obtain

$$(4.10) \quad \begin{aligned} S(Y, U) - S(Y, V) &= (a+b)A(Y) + dB(Y) - (a+c)B(Y) \\ &\quad - dA(Y) - eD(Y, V). \end{aligned}$$

Equating (4.9) and (4.10), we get

$$(4.11) \quad \begin{aligned} k(n-1)[A(Y) - B(Y)] &= (a+b-d)A(Y) + B(Y)(d-a-c) \\ &\quad - eD(Y, V). \end{aligned}$$

Putting $Y = U$ in (4.11), we obtain

$$(4.12) \quad k = \frac{a+b-d}{n-1}.$$

And also putting $Y = V$ in (4.11), we obtain

$$(4.13) \quad k = \frac{a+c+\tilde{m}-d}{n-1},$$

where, $eD(V, V) = \tilde{m}$ (say). Equating (4.12) and (4.13), we get $b = c + \tilde{m}$. So, $k = \frac{a+b-d}{n-1}$. Now,

$$(4.14) \quad S(Y, U) = (a+b-d)g(Y, U),$$

and

$$(4.15) \quad S(Y, V) = (a + b - d)g(Y, V).$$

Therefore we can say that

Theorem 4.2. *In a $N(k) - MS(QE)_n$ manifold, the orthogonal vector fields U and V are the eigen vectors corresponding to the eigen value $(a + b - d)$.*

5. RICCI SEMI-SYMMETRIC $N(k) - MS(QE)_n$

A $N(k) - MS(QE)_n$ is said to be Ricci semi symmetric manifold [7] if it satisfy $R(X, Y).S = 0, \forall X, Y$ where $R(X, Y)$ denotes the curvature operator. Then we have,

$$(5.1) \quad S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0.$$

Putting $X = U$ in (5.1),

$$(5.2) \quad k[g(Y, Z)S(U, W) - A(Z)S(Y, W) + g(Y, W)S(Z, U) - A(W)S(Z, Y)] = 0.$$

Putting $W = U$ in (5.2), we get

$$(5.3) \quad k[g(Y, Z)S(U, U) - A(Z)S(Y, U) + g(Y, U)S(Z, U) - A(U)S(Z, Y)] = 0.$$

That is,

$$(5.4) \quad k [(a + b)g(Y, Z) - A(Z)\{ag(Y, U) + bA(Y)A(U) + cB(Y)(U) + d(A(Y)B(U) + A(U)B(Y)) + eD(Y, U)\} + A(Y)\{ag(Z, U) + bA(Z)A(U) + cB(Z)(U) + d(A(Z)B(U) + A(U)B(Z)) + eD(Z, U)\} - S(Y, Z)] = 0.$$

From the above equation, we obtain

$$(5.5) \quad k[(a + b)g(Y, Z) + d\{A(Y)B(Z) - A(Z)B(Y)\} - S(Y, Z)] = 0.$$

If $k \neq 0$, then (M^n, g) becomes $M(QE)_n$. Therefore, we must have $k = 0$.

Conversely suppose $k = 0$. Then we obtain $R(U, X)Y = 0$ which implies $R(U, X).S = 0$. Thus we have,

Theorem 5.1. *A $N(k) - MS(QE)_n$ manifold satisfies $R(U, X).S = 0$ if and only if $a + b - d = 0$.*

6. RICCI RECURRENT $N(k) - MS(QE)_n$

A $N(k) - MS(QE)_n$ manifold is called *Ricci recurrent* if it satisfies

$$(6.1) \quad (\nabla_X S)(Y, Z) = T(X)S(Y, Z),$$

where $T(X)$ is a non-zero 1-form. But,

$$(6.2) \quad (\nabla_X S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

That is,

$$(6.3) \quad T(X)S(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

Putting $Y = Z = U$ in (6.3), we get

$$(6.4) \quad (a + b)T(X) = X(a + b) - S(\nabla_X U, U) - S(U, \nabla_X U),$$

or,

$$(6.5) \quad (a + b)T(X) = X(a + b) - 2[bA(\nabla_X U) + dB(\nabla_X U)],$$

or,

$$(6.6) \quad (a + b)T(X) = X(a + b) - 2[dB(\nabla_X U)].$$

So,

$$(a + b)T(X) = X(a + b) \iff B(\nabla_X U) = 0.$$

But, $B(\nabla_X U) = 0$ implies either $\nabla_X U \perp V$ or U is a parallel vector field.

Similarly, if we put $Y = Z = V$ in (6.3), we obtain

$$(6.7) \quad \begin{aligned} (a + b + eD(V, V))T(X) &= X(a + b + eD(V, V)) - S(\nabla_X V, V) \\ &\quad - S(V, \nabla_X V), \end{aligned}$$

or,

$$(6.8) \quad \begin{aligned} (a + b + eD(V, V))T(X) &= X(a + b + eD(V, V)) - 2[cB(\nabla_X V) \\ &\quad + A(\nabla_X V) + eD(\nabla_X V, V)]. \end{aligned}$$

So, $(a + b + \tilde{m})T(X) = X(a + b + \tilde{m}) - 2eD(\nabla_X V, V)$ iff $A(\nabla_X V) = 0$.

But, $A(\nabla_X V) = 0$ implies either $\nabla_X V \perp U$ or V is a parallel vector field, where, $eD(V, V) = \tilde{m}$.

Thus, we can say that

Theorem 6.1. *A $N(k) - MS(QE)_n$ manifold is Ricci recurrent, then either the vector field U (or V) is parallel or $\nabla_X U \perp V$ (or $\nabla_X V \perp U$).*

7. GENERALIZED RICCI RECURRENT $N(k) - MS(QE)_n$

A $N(k) - MS(QE)_n$ manifold is called *generalized Ricci recurrent* [5] if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$(7.1) \quad (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(X)g(Y, Z),$$

where $\alpha(X)$ and $\beta(X)$ are two nowhere vanishing 1-forms such that $\alpha(X) = g(X, \rho)$ and $\beta(X) = g(X, \mu)$; ρ and μ being associated vector fields of the 1-forms α and β , respectively.

Definition. A Riemannian manifold is said to *admit cyclic parallel Ricci tensor* if

$$(7.2) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

Now, we prove the following:

Theorem 7.1. *On a generalized Ricci recurrent $N(k) - MS(QE)_n$ with cyclic parallel Ricci tensor, the Ricci tensor is of the form*

$$(7.3) \quad \begin{aligned} \alpha(U)S(X, Y) = & -\beta(U)g(X, Y) - (a + b)[\alpha(X)A(Y) + \alpha(Y)A(X)] \\ & - d[\alpha(X)\beta(Y) + \alpha(Y)\beta(X)] \\ & - [A(X)\beta(Y) + A(Y)\beta(X)]. \end{aligned}$$

Proof. Suppose that M is a generalized Ricci recurrent $N(k) - MS(QE)_n$ admitting cyclic parallel Ricci tensor. Then using (7.1) in (7.2), we get

$$(7.4) \quad \begin{aligned} & \alpha(X)S(Y, Z) + \beta(X)g(Y, Z) + \alpha(Y)S(Z, X) \\ & + \beta(Y)g(Z, X) + \alpha(Z)S(X, Y) \\ & + \beta(Z)g(X, Y) = 0. \end{aligned}$$

Setting $Z = U$ in (7.4), Using (1.9) and (4.7), we get the relation (7.3). Contracting (7.3) over X and Y , we get

$$(7.5) \quad \begin{aligned} \alpha(U)r = & -n\beta(U) - 2(a + b)\alpha(U) - 2\beta(U) \\ & - 2d\alpha(V). \end{aligned}$$

□

This leads to the following:

Corollary. *On a generalized Ricci recurrent $N(k)$ -mixed super quasi Einstein manifold with cyclic parallel Ricci tensor, the scalar curvature is of the form (7.5).*

We now consider a generalized Ricci recurrent $N(k) - MS(QE)_n$ whose Ricci tensor is of Codazzi type. Then we have ([8], [10])

$$(7.6) \quad (\nabla_X S)(Y, Z) = (\nabla_Z S)(X, Y).$$

Using (7.1) in (7.6), we obtain

$$(7.7) \quad \alpha(X)S(Y, Z) + \beta(X)g(Y, Z) = \alpha(Z)S(X, Y) + \beta(Z)g(X, Y).$$

Setting $Z = U$ in (7.7), using (1.9) and (4.7), we get the relation

$$(7.8) \quad \begin{aligned} \alpha(U)S(X, Y) = & \beta(U)g(X, Y) - d\alpha(X)\beta(Y) \\ & + [\alpha(X)(a + b) + \beta(X)]A(Y). \end{aligned}$$

This leads to the following:

Theorem 7.2. *On a generalized Ricci recurrent $N(k) - MS(QE)_n$ whose Ricci tensor is of Codazzi type, the Ricci tensor is of the form (7.8).*

Also from (7.8), we can state the following:

Corollary. *On a generalized Ricci recurrent $N(k) - MS(QE)_n$ whose Ricci tensor is of Codazzi type, the scalar curvature is given by*

$$(7.9) \quad \alpha(U)r = -(n - 1)\beta(U) - d\alpha(V) + \alpha(U)(a + b).$$

8. $N(k) - MS(QE)_n$ SATISFYING THE CONDITION $\tilde{C}(U, X).S = 0$

The *concircular curvature tensor* $\tilde{C}$ of type $(1, 3)$ of n -dimensional *Riemannian manifold* (M^n, g) , $(n \geq 3)$ is defined by [13]

$$(8.1) \quad \tilde{C}(X, Y)W = R(X, Y)W - \frac{r}{n(n-1)}[g(Y, W)X - g(X, W)Y]$$

for any vector fields $X, Y, Z \in \chi(M)$. Let us consider a $N(k) - MS(QE)_n$ $(n \geq 3)$ satisfying the condition $(\tilde{C}(U, X).S)(Y, Z) = 0$.

Putting $Z = U$, we have

$$(8.2) \quad S(\tilde{C}(U, X)Y, U) + S(Y, \tilde{C}(U, X)U) = 0.$$

Now using the definition of k -nullity distribution in (8.1), we obtain

$$(8.3) \quad \tilde{C}(U, X)Y = [k - \frac{r}{n(n-1)}][g(X, Y)U - A(Y)X]$$

and

$$(8.4) \quad \tilde{C}(U, X)U = [k - \frac{r}{n(n-1)}][A(X)U - X].$$

Now,

$$(8.5) \quad \begin{aligned} S(\tilde{C}(U, X)Y, U) &= [k - \frac{r}{n(n-1)}][g(X, Y)(a + b) - (a + b)A(X)A(Y) \\ &\quad - dB(X)A(Y)] \end{aligned}$$

and

$$(8.6) \quad \begin{aligned} S(Y, \tilde{C}(U, X)U) &= [k - \frac{r}{n(n-1)}][(a + b)A(X)A(Y) + dA(X)B(Y) \\ &\quad - S(X, Y)]. \end{aligned}$$

From (8.2), we get

$$(8.7) \quad \begin{aligned} [k - \frac{r}{n(n-1)}][g(X, Y)(a + b) + d\{A(X)B(Y) - B(X)A(Y)\} \\ - S(X, Y)] = 0. \end{aligned}$$

So, either scalar curvature $r = kn(n-1)$ or (M^n, g) becomes $M(QE)_n$. But (M^n, g) is not $M(QE)_n$. So, we can state the following:

Theorem 8.1. *The $N(k) - MS(QE)_n$ satisfying the condition $\tilde{C}(U, X).S = 0$, i.e., concircularly Ricci symmetric iff its scalar curvature is $n(a + b - d)$.*

9. $N(k) - MS(QE)_n$ SATISFYING THE CONDITION $\tilde{P}(U, X).S = 0$

The *Weyl projective curvature tensor* $\tilde{P}$ of type $(1, 3)$ of n -dimensional *Riemannian manifold* (M^n, g) , $(n \geq 3)$ is defined by [13]

$$(9.1) \quad \tilde{P}(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]$$

for any vector fields $X, Y, Z \in \chi(M)$. Let us consider a $MS(QE)_n$ ($n \geq 3$) satisfying the condition $(\tilde{P}(U, X).S)(Y, Z) = 0$.

Putting $Z = U$, we have

$$(9.2) \quad S(\tilde{P}(U, X)Y, U) + S(Y, \tilde{P}(U, X)U) = 0.$$

Now using the definition of k -nullity distribution in (9.1), we obtain

$$(9.3) \quad \begin{aligned} S(\tilde{P}(U, X)Y, U) &= k[g(X, Y)(a + b) - A(Y)\{(a + b)A(X) + dB(X)\}] \\ &\quad - \frac{1}{n-1}[S(X, Y)(a + b) - S(Y, U)S(X, U)] \end{aligned}$$

and

$$(9.4) \quad \begin{aligned} S(Y, \tilde{P}(U, X)U) &= k[A(X)\{(a + b)A(Y) + dB(Y)\} - S(X, Y)] \\ &\quad - \frac{1}{n-1}[S(X, U)S(Y, U) - (a + b)S(X, Y)]. \end{aligned}$$

From (9.2),

$$(9.5) \quad \begin{aligned} k[g(X, Y)(a + b) + d\{A(X)B(Y) - B(X)A(Y)\} \\ - S(X, Y)] = 0. \end{aligned}$$

So, $k = 0$, otherwise (M^n, g) becomes $M(QE)_n$. Thus we have

Theorem 9.1. *The $N(k) - MS(QE)_n$ satisfying the condition $\tilde{P}(U, X).S = 0$, i.e., Weyl projectively Ricci symmetric iff $k = 0$.*

10. RICCI-PSEUDOSYMMETRIC $N(k) - MS(QE)_n$

An n -dimensional Riemannian manifold (M^n, g) is called *Ricci-pseudosymmetric* [6] if the tensors $R.S$ and $Q(g, S)$ are linearly dependent, where

$$(10.1) \quad (R(X, Y).S)(Z, W) = -S(R(X, Y)Z, W) - S(Z, R(X, Y)W),$$

$$(10.2) \quad Q(g, S)(Z, W; X, Y) = -S((X \wedge Y)Z, W) - S(Z, (X \wedge Y)W)$$

and

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$$

for vector fields X, Y, Z, W on M^n , R denotes the curvature tensor of M^n . The condition of *Ricci-pseudosymmetry* is equivalent to the relation

$$(10.3) \quad (R(X, Y).S)(Z, W) = L_S Q(g, S)(Z, W; X, Y)$$

holding on the set

$$(10.4) \quad U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\},$$

where L_S is some function on U_S . If $R.S = 0$ then M^n is called *Ricci-semisymmetric*. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true [6].

Assume that M^n is Ricci-pseudosymmetric. Then by the use of (10.1) to (10.4), we can obtain

$$(10.5) \quad \begin{aligned} S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = & L_S\{g(Y, Z)S(X, W) \\ & - g(X, Z)S(Y, W) + g(Y, W)S(X, Z) \\ & - g(X, W)S(Y, Z)\}. \end{aligned}$$

Since M^n is also $N(k) - MS(QE)_n$, using the properties of the curvature tensor R we get

$$(10.6) \quad \begin{aligned} & b[A(R(X, Y)Z)A(W) + A(Z)A(R(X, Y)W)] + c[B(R(X, Y)Z)B(W) \\ & + B(Z)B(R(X, Y)W)] + d[A(R(X, Y)Z)B(W) + A(W)B(R(X, Y)Z) \\ & + A(Z)B(R(X, Y)W) + A(R(X, Y)W)B(Z)] + e[D(R(X, Y)Z, W) \\ & + D(Z, R(X, Y)W)] = L_S\{b[g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\ & + g(Y, W)A(X)A(Z) - g(X, W)A(Y)A(Z)] + c[g(Y, Z)B(X)B(W) \\ & - g(X, Z)B(Y)B(W) + g(Y, W)B(X)B(Z) - g(X, W)B(Y)B(Z)] \\ & + d[g(Y, Z)A(X)B(W) + g(Y, Z)A(W)B(X) - g(X, Z)A(Y)B(W) \\ & - g(X, Z)A(W)B(Y) + g(Y, W)A(X)B(Z) + g(Y, W)A(Z)B(X) \\ & - g(X, W)A(Y)B(Z) - g(X, W)A(Z)B(Y)] + e[g(Y, Z)D(X, W) \\ & - g(X, Z)D(Y, W) + g(Y, W)D(X, Z) - g(X, W)D(Y, Z)]\}. \end{aligned}$$

Putting $Y = Z = U$ in (10.6), we get

$$(10.7) \quad \begin{aligned} & kb[A(X)A(W) - g(X, W)] + kc[B(X)B(W)] + 2kd[A(W)B(X)] \\ & + ke[D(X, W)] = L_S\{b[A(X)A(W) - g(X, W)] \\ & + c[B(X)B(W)] + 2d[A(W)B(X)] + eD(X, W)\}. \end{aligned}$$

Putting $X = W = e_i$ in (10.7) where $\{e_i\}$, $i = 1, 2, \dots, n$ be an orthonormal basis of the tangent space at any point of the manifold and taking summation over i , $1 \leq i \leq n$, we obtain

$$(10.8) \quad L_S = k = \frac{a + b - d}{n - 1}.$$

Thus we can state the following:

Theorem 10.1. *The Ricci-pseudosymmetric $N(k) - MS(QE)_n$ is Ricci semi symmetric manifold iff $a + b - d = 0$.*

11. EXAMPLE OF A 4-DIMENSIONAL $N(k) - MS(QE)_n$

Here we construct a nontrivial concrete example of a $N(k) - MS(QE)_n$. Let us consider a Riemannian metric g on the 4-dimensional real number space M^4 by

$$(11.1) \quad ds^2 = g_{ij}dx^i dx^j = (e^{x^2})(dx^1)^2 + (x^1 x^3)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2,$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 . Then the only non vanishing components of the Christoffel symbols, the curvature tensors and the Ricci tensor are

$$(11.2) \quad [12, 1] = \frac{e^{x^2}}{2}, \quad [21, 2] = x^1(x^3)^2, \quad [23, 2] = (x^1)^2x^3$$

$$(11.3) \quad R_{1223} = R_{3221} = x^1x^3, \quad R_{1221} = \frac{e^{x^2}}{4}, \quad R_{2113} = R_{3112} = -\frac{e^{x^2}}{2x^3}$$

$$(11.4) \quad R_{11} = \frac{e^{x^2}}{4(x^1x^3)^2}, \quad R_{22} = \frac{1}{4}, \quad R_{13} = \frac{1}{x^1x^3}, \quad R_{23} = -\frac{1}{2x^3}$$

and the components which can be obtained from these by symmetric properties. So, M^4 is a *Riemannian manifold* of non-vanishing scalar curvature. We shall now show that this manifold is an $N(k) - MS(QE)_4$. Let us now define

$$a = \frac{1}{8(x^1x^3)^2}, \quad b = \frac{1}{(x^1x^3)^2}, \quad c = \frac{1}{4(x^1x^3)^2}, \quad d = -\frac{\sqrt{e^{x^2}}}{(x^1)^3(x^3)^4}, \quad e = -\frac{1}{(x^1x^3)^2}$$

and the 1-forms are

$$A_i(x) = \begin{cases} \frac{(x^1x^3)^2}{8}, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$B_i(x) = \begin{cases} \sqrt{e^{x^2}}, & \text{if } i = 1 \\ \frac{(x^1x^3)^2}{8}, & \text{if } i = 3 \\ 0, & \text{otherwise} \end{cases}$$

and the associated tensor as

$$D_{ij}(x) = \begin{cases} \frac{e^{x^2}}{4}, & \text{if } i = j = 1 \\ 0, & \text{otherwise} \end{cases}$$

then we have

- (i) $R_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + 2dA_1B_1 + eD_{11}$,
- (ii) $R_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + 2dA_2B_2 + eD_{22}$,
- (iii) $R_{13} = ag_{13} + bA_1A_3 + cB_1B_3 + d(A_1B_3 + A_3B_1) + eD_{13}$,
- (iv) $R_{23} = ag_{23} + bA_2A_3 + cB_2B_3 + d(A_2B_3 + A_3B_2) + eD_{23}$.

Since all the cases other than (i) – (iv) are trivial, we can say

$$R_{ij} = ag_{ij} + bA_iA_j + cB_iB_j + d(A_iB_j + A_jB_i) + eD_{ij}, \text{ for } i, j = 1, 2, 3, 4.$$

So, we can say that the manifold under consideration is an $N(k) - MS(QE)_4$, where $k = \frac{9x^1(x^3)^2 + 8\sqrt{e^{x^2}}}{24(x^1)^3(x^3)^4}$. Thus if (M^4, g) is a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (e^{x^2})(dx^1)^2 + (x^1x^3)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2,$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 , then it is an $N(k) - MS(QE)_4$ with nonzero and nonconstant scalar curvature.

REFERENCES

- [1] A. Bhattacharyya, M. Tarafdar, and D. Debnath. On mixed super quasi-Einstein manifolds. *Differ. Geom. Dyn. Syst.*, 10:44–57, 2008.
- [2] M. C. Chaki. On super quasi Einstein manifolds. *Publ. Math. Debrecen*, 64(3-4):481–488, 2004.
- [3] M. C. Chaki and R. K. Maity. On quasi-Einstein manifolds. *Publ. Math. Debrecen*, 57:297–306, 2000.
- [4] U. C. De and G. C. Ghosh. On generalized quasi-Einstein manifolds. *Kyungpook Math. J.*, 44:607–615, 2004.
- [5] U. C. De, N. Guha, and D. Kamilya. On generalized Ricci-recurrent manifolds. *Tensor. New series*, 56(3):312–317, 1995.
- [6] R. Deszcz. On pseudosymmetric spaces. *Bull. Soc. Math. Belg. Sér. A*, 44(1):1–34, 1992.
- [7] R. Deszcz and M. Hotlos. Remarks on Riemannian manifolds satisfying certain curvature condition imposed on the Ricci tensor. *Prace Nauk. Pol. Szczec*, 11:23–34, 1989.
- [8] D. Ferus. A remark on codazzi tensors in constant curvature spaces. In *Global differential geometry and global analysis*, pages 257–257. Springer, 1981.
- [9] H. Nagaraja. On $N(k)$ -mixed quasi Einstein manifolds. *European Journal of Pure and Applied Mathematics*, 3(1):16–25, 2009.
- [10] J. A. Schouten. Ricci-calculus: An introduction to tensor analysis and its geometrical applications. *Verlag Berlin Heidelberg GmbH, Springer*, 1954.
- [11] S. Tanno. Ricci curvatures of contact Riemannian manifolds. *Tohoku Mathematical Journal, Second Series*, 40(3):441–448, 1988.
- [12] M. M. Tripathi and J. Kim. On $N(k)$ -quasi Einstein manifolds. *Communications-Korean Mathematical Society*, 22(3):411, 2007.
- [13] K. Yano and M. Kon. *Structures on manifolds*, volume 3. World scientific, 1985.

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