

ON $N(k)$ MIXED QUASI EINSTEIN MANIFOLDS AND SOME GLOBAL PROPERTIES

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This paper has been dedicated to Prof. M. Majumdar and Prof. A. Bhattacharyya

ABSTRACT. In this paper we have studied quasi conformally flat, conharmonically flat and projectively flat $N(k)$ -mixed quasi Einstein manifold, Ricci-semi symmetric $N(k)$ -mixed quasi Einstein manifold $N(k)-(MQE)_n$, ($n > 3$) and studied some properties on it.

1. INTRODUCTION

The notion of quasi Einstein manifold was introduced in a paper [8] by M. C. Chaki and R. K. Maity. According to them a non-flat Riemannian manifold $(M^n, g), (n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$(1) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y)$$

and is not identically zero, where a, b are scalars $b \neq 0$ and A is a non-zero 1-form such that

$$(2) \quad g(X, U) = A(X), \quad \forall X \in TM.$$

U being a unit vector field. In such a case a, b are called the associated scalars. A is called the associated 1-form and U is called the generator of the manifold. Such an n -dimensional manifold is denoted by the symbol $(QE)_n$.

Again, U. C. De and G. C. Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$(3) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$$

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where a, b, c , are non-zero scalars and A, B are two 1-forms such that

$$(4) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X)$$

U, V being unit vectors which are orthogonal, i.e.

$$(5) \quad g(U, V) = 0.$$

The vector fields U and V are called the generators of the manifold. This type of manifold are denoted by $G(QE)_n$.

The k -nullity distribution [15] of a Riemannian manifold M is defined by

$$(6) \quad N(k) : p \rightarrow N_p(k) = \{Z \in T_p M \setminus R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)\}.$$

for all $X, Y \in TM$ and k is a smooth function. M. M. Tripathi and Jeong Jik Kim [14] introduced the notion of $N(k)$ -quasi Einstein manifold which defined as follows: If the generator U belongs to the k -nullity distribution $N(k)$, then a quasi Einstein manifold (M^n, g) is called $N(k)$ -quasi Einstein manifold.

In [13], H. G. Nagaraja introduced the concept of $N(k)$ -mixed quasi Einstein manifold and mixed quasi constant curvature. A non flat Riemannian manifold (M^n, g) is called a $N(k)$ -mixed quasi Einstein manifold if its Ricci tensor of type $(0, 2)$ is non zero and satisfies the condition

$$(7) \quad S(X, Y) = ag(X, Y) + bA(X)B(Y) + cB(X)A(Y),$$

where a, b, c , are smooth functions and A, B are non zero 1-forms such that

$$(8) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X) \quad \forall X,$$

U, V being the orthogonal unit vector fields called generators of the manifold belong to $N(k)$. Such a manifold is denoted by the symbol $N(k) - (MQE)_n$.

Again a Riemannian manifold (M^n, g) is called of mixed quasi constant curvature if it is conformally flat and curvature tensor R of type $(0, 4)$ satisfies the condition

$$(9) \quad \begin{aligned} R(X, Y, Z, W) = & p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + q[g(X, W)A(Y)B(Z) - g(X, Z)A(Y)B(W) \\ & + g(X, W)A(Z)B(Y) - g(X, Z)A(W)B(Y)] \\ & + s[g(Y, Z)A(W)B(X) - g(Y, W)A(Z)B(X) \\ & + g(Y, Z)A(X)B(W) - g(Y, W)A(X)B(Z)]. \end{aligned}$$

2. PRELIMINARIES

We know in a n -dimensional ($n > 2$) Riemannian manifold the covariant quasi conformal curvature tensor is defined as ([1], [3], [7], [11])

$$(10) \quad \begin{aligned} \tilde{C}(X, Y, Z, W) = & \acute{a}\acute{R}(X, Y, Z, W) + \acute{b}[S(Y, Z)g(X, W) \\ & - S(X, Z)g(Y, W) + g(Y, Z)g(QX, W) - g(X, W)g(QY, W)] \\ & - \frac{r}{n}[\frac{\acute{a}}{n-1} + 2\acute{b}][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \end{aligned}$$

where

$$(11) \quad g(C(X, Y)Z, W) = \tilde{C}(X, Y, Z, W).$$

$$(12) \quad g(R(X, Y)Z, W) = \tilde{R}(X, Y, Z, W).$$

The projective curvature tensor is denoted by $\tilde{P}(X, Y, Z, W)$ and in a V_n ($n > 2$) it is defined as

$$(13) \quad \begin{aligned} &\tilde{P}(X, Y, Z, W) \\ &= R(X, Y, Z, W) - \frac{1}{n-1} [S(Y, Z)g(X, W) - S(Y, W)g(X, Z)]. \end{aligned}$$

From (7) and (8), we get

$$(14) \quad S(X, X) = a|X|^2 + (b+c)|g(X, U)g(X, V)|, \quad \forall X.$$

Let θ_1 be the angle between U and any vector X ; θ_2 be the angle between V and any vector X . Then

$$\cos \theta_1 = \frac{g(X, U)}{\sqrt{g(U, U)}\sqrt{g(X, X)}} = \frac{g(X, U)}{\sqrt{g(X, X)}} \quad (\text{as } g(U, U) = 1)$$

and $\cos \theta_2 = \frac{g(X, V)}{\sqrt{g(X, X)}}$. If $b > 0$ and $c > 0$ we have from (14)

$$(15) \quad (a+b+c)|X|^2 \geq a|X|^2 + (b+c)|g(X, U)g(X, V)| = S(X, X)$$

Now, contracting (7) over X and Y , we get

$$(16) \quad r = na$$

where r is the scalar curvature.

Again from (7) we have

$$(17) \quad S(U, U) = a$$

$$(18) \quad S(V, V) = a$$

If X is a unit vector field, then $S(X, X)$ is the Ricci-curvature in the direction of X .

Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S , where

$$(19) \quad g(QX, Y) = S(X, Y) \quad \forall X, Y \in TM.$$

Let l^2 denote the squares of the lengths of the Ricci-tensor S . Then

$$(20) \quad l^2 = \sum_{i=1}^n S(Qe_i, e_i)$$

where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at a point of $N(k) - (MQE)_n$.

Now from (7) we get

$$(21) \quad S(Qe_i, e_i) = ag(Qe_i, e_i)bA(Qe_iB(e_i)) + cB(Qe_i)A(e_i)$$

i.e. $l^2 = na^2 + b^2 + c^2$.

3. QUASI CONFORMALLY FLAT $N(k)$ - MIXED QUASI EINSTEIN MANIFOLD

Let a $N(k)$ - mixed quasi Einstein manifold is quasi conformally flat. Considering $\tilde{C}(X, Y)Z = 0$. For all vector fields X, Y, Z it follows from (10) that

$$(22) \quad R(X, Y)Z = \frac{r}{na} \left[\frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y] \\ - \frac{b}{a} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY],$$

or

$$(23) \quad g(R(X, Y)Z, W) = \frac{r}{na} \left[\frac{a}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ - \frac{b}{a} [S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ + g(Y, Z)g(QX, W) - g(X, Z)g(QY, W)].$$

Using (7), (8), (12) in (22) we get

$$(24) \quad \hat{R}(X, Y, Z, W) = \left\{ \frac{r}{na} \left[\frac{a}{n-1} + 2b \right] - \frac{2ab}{a} \right\} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ - \frac{bb}{a} [g(X, W)A(Y)B(Z) - g(Y, W)A(X)B(Z) \\ + g(Y, Z)A(X)B(W) - g(X, Z)A(Y)B(W)] \\ - \frac{cc}{a} [g(X, W)B(Y)A(Z) - g(Y, W)B(X)A(Z) \\ + g(Y, Z)B(X)A(W) - g(X, Z)B(Y)A(W)].$$

Thus from (24) we get

Theorem 3.1. *A quasi conformally flat $N(k) - (MQE)_n$ is a manifold of mixed quasi constant curvature.*

Corollary 3.1. *A conharmonically flat $N(k) - (MQE)_n$ is a manifold of mixed quasi constant curvature.*

Corollary 3.2. *A projectively flat $N(k) - (MQE)_n$ is not a manifold of mixed quasi constant curvature.*

4. RICCI SEMI-SYMMETRIC $N(k) - (MQE)_n (n > 3)$

Chaki and Maity proved that $(QE)_n (n > 3)$ is Ricci Semi-symmetric if and only if $A(R(X, Y)Z) = 0$. Let us suppose that $N(k) - (MQE)_n (n > 3)$ is Ricci-Semi symmetric. Then

$$(25) \quad A(R(X, Y)Z) = 0.$$

From (25) we get

$$(26) \quad A(Q(X)) = 0$$

where Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S . Then

$$(27) \quad g(QX, Y) = S(X, Y).$$

Then from (7) we get

$$(28) \quad A(Q(X)) = aA(X) + cB(X).$$

From (26) and (28) it follows that

$$(29) \quad aA(X) + cB(X) = 0.$$

Thus we can state the following

Theorem 4.1. *If a $N(k) - (MQE)_n$ is Ricci Semi symmetric than $aA(X) + cB(X) = 0$.*

5. $N(k) - (MQE)_n (n > 3)$ WITH DIVERGENCE FREE QUASI CONFORMAL CURVATURE TENSOR

We know quasi conformal curvature tensor is said to be conservative if divergence of $\acute{C}$ vanishes, i.e. $\text{div } \acute{C} = 0$. In this section we obtain a sufficient condition for a $N(k) - (MQE)_n$ be a quasi conformally conservative. In a $N(k) - (MQE)_n$ if a, b and c are constant, then contracting (7) we obtain

$$(30) \quad r = na, \text{ i.e. } dr = 0$$

where r is the scalar curvature. Using (30) in (10) we get

$$(31) \quad (\nabla_W \acute{C})(X, Y, Z) = a_1(\nabla_W R)(X, Y)Z \\ + b_1[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y + g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y].$$

We know that

$$(32) \quad (\text{div } R)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z).$$

Now from (7) we get

$$(33) \quad (\nabla_X S)(Y, Z) = b[(\nabla_X A)(Y)B(Z) + (\nabla_X B)(Z)A(Y)] \\ + c[(\nabla_X B)(Y)A(Z) + (\nabla_X A)(X)B(Z)]$$

where b and c are constant. Hence contracting (31) and using (33) we obtain

$$\begin{aligned}
 (\operatorname{div} C)(X, Y, Z) &= 2b(a_1 + b_1)[(\nabla_X A)(Y)B(Z) + (\nabla_X B)(Z)A(Y) \\
 &\quad - (\nabla_Y A)(X)B(Z) - (\nabla_Y B)(Z)A(X)] \\
 &\quad + 2c(a_1 + b_1)[(\nabla_X B)(Y)A(Z) + (\nabla_X A)(Z)B(Y) \\
 (34) \quad &\quad - (\nabla_Y B)(X)A(Z) - (\nabla_Y A)(Z)B(X)] \\
 &\quad + bb_1[(\nabla_U A)(X) + B(X) \operatorname{div} U + (\nabla_U B)(X) + A(X) \operatorname{div} U]g(Y, Z) \\
 &\quad - cb_1[(\nabla_U A)(Y) + B(Y) \operatorname{div} U + (\nabla_U B)(Y) + A(Y) \operatorname{div} U]g(X, Z).
 \end{aligned}$$

Now if we consider the generator U of the manifold is a recurrent vector field [6] with associated 1-form A , not being the 1-form of recurrence, gives $\nabla_X U = D(X)U$, where D is the 1-form of recurrence, we get

$$(35) \quad g(\nabla_X U, Y) = g(D(X)U, Y), \text{ i.e. } (\nabla_X A)(Y) = D(X)A(Y)$$

so we get

$$\begin{aligned}
 (\operatorname{div} \dot{C})(X, Y, Z) &= 2b(a_1 + b_1)[D(X)A(Y)B(Z) + D(X)B(Z)A(Y) \\
 &\quad - D(Y)A(X)B(Z) - D(Y)B(Z)A(X)] \\
 &\quad + 2c(a_1 + b_1)[D(X)B(Y)A(Z) + D(X)A(Z)B(Y) \\
 (36) \quad &\quad - D(Y)B(X)A(Z) - D(Y)A(Z)B(X)] \\
 &\quad + bb_1[D(U)A(X) + D(U)B(X)]g(Y, Z) \\
 &\quad - cb_1[D(U)A(Y) + D(U)B(Y)]g(X, Z).
 \end{aligned}$$

Since $(\nabla_X A)(U) = 0$, it follows from (35) we get $D(X) = 0$. Hence from (36) we get $(\operatorname{div} \dot{C})(X, Y, Z) = 0$. Thus we get

Theorem 5.1. *If in a $N(k) - (MQE)_n (n > 3)$ the associated scalars are constants and generator U of the manifold is a recurrent vector field with the associated 1-form A not being the 1-form of recurrence, then the manifold is quasi-conformally conservative.*

6. SUFFICIENT CONDITION FOR A COMPACT, ORIENTABLE

$N(k) - (MQE)_n (n \geq 3)$ WITHOUT BOUNDARY TO BE ISOMETRIC TO A SPHERE

In this section we consider a compact, orientable $N(k) - (MQE)_n$ without boundary having constant associated scalars a, b, c . Then from (16) and (19), it follows that the scalar curvature is constant and so also is the length of the Ricci-tensor.

We further suppose that $N(k) - (MQE)_n$ under consideration admits a non-isometric conformal motion generated by a vector field X . Since l^2 is constant, it follows that

$$(37) \quad \mathcal{L}_X l^2 = 0.$$

where $\mathcal{L}_X$ denotes Lie differentiation with respect to X . Now, it is known ([2], [4], [5], [9], [12]) that if a compact Riemannian manifold M of dimension $n > 2$ with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\mathcal{L}_X l^2 = 0$ then M is isometric to a sphere. But a sphere is Einstein so that b and c vanish which is a contradiction. This leads to the following theorem.

Theorem 6.1. *A compact orientable $N(k)$ -mixed quasi Einstein manifold $N(k) - (MQE)_n (n \geq 3)$ without boundary does not admit a non-isometric conformal vector field.*

7. KILLING VECTOR FIELD IN A COMPACT ORIENTABLE $N(k) - (MQE)_n (n \geq 3)$ WITHOUT BOUNDARY

In this section, we consider a compact, orientable $N(k) - (MQE)_n (n \geq 3)$ without boundary with a, b, c as associated scalars and U and V as the generators.

It is known [4] that in such a manifold M , the following relation holds

$$(38) \quad \int_M [S(X, X) - |\nabla X|^2 - (\operatorname{div} X)^2] dv \leq 0 \quad \forall X.$$

If X is a Killing vector field, then $\operatorname{div} X = 0$ [4]. Hence (38) takes the form

$$(39) \quad \int_M [S(X, X) - |\nabla X|^2] dv = 0.$$

Let $b > 0, c > 0$ then by (15)

$$(40) \quad (a + b + c)|X|^2 \geq S(X, X)$$

Therefore,

$$(41) \quad (a + b + c)|X|^2 - |\nabla X|^2 \geq S(X, X) - |\nabla X|^2$$

Consequently,

$$(42) \quad \int_M [(a + b + c)|X|^2 - |\nabla X|^2] dv \geq \int_M [S(X, X) - |\nabla X|^2] dv$$

and by (39)

$$(43) \quad \int_M [(a + b + c)|X|^2 - |\nabla X|^2] dv \geq 0$$

If $a + b + c < 0$, then

$$(44) \quad \int_M [(a + b + c)|X|^2 - |\nabla X|^2] dv = 0.$$

Therefore, $X = 0$. This leads to the following theorem.

Theorem 7.1. *If in a compact, orientable $N(k) - (MQE)_n (n \geq 3)$ without boundary the associated scalars are such that $b > 0, c > 0$ and $a + b + c < 0$ then there exists no non-zero killing vector field in this manifold.*

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