

QUASI-CONFORMALLY FLAT AND PROJECTIVELY FLAT TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. The object of the present paper is to study quasi-conformally flat and projectively flat contact manifolds. We have also studied Quasi-conformally flat and projectively flat trans-Sasakian manifolds. We obtained condition for trans-Sasakian manifold to be quasi-conformally flat and projectively flat. The value of scalar curvature has been obtained in quasi-conformally flat and projectively flat trans-Sasakian manifolds.

1. INTRODUCTION

Oubina [11] introduced a manifold which generalizes both α -Sasakian and β -Kenmotsu manifolds such manifold was called a trans-Sasakian manifold of type (α, β) . Sasakian, Kenmotsu, and cosymplectic manifold are particular cases of trans-Sasakian manifolds. Trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic [2], α -Sasakian ([3], [14]) and β -Kenmotsu ([3], [8]) respectively. Concept of nearly trans-Sasakian manifold was introduced by C. Gherghe [7]. Marrero [9] constructed a three dimensional trans-Sasakian manifold. Prasad and Srivastava [12] obtained certain results on trans-Sasakian manifolds. Jeong- Sik kim et al. [7] studied a generalized Ricci-recurrent trans-Sasakian manifolds. A quasi-conformal curvature tensor $\check{C}$ was defined by Yano and Sawaki [17] as follows:

$$(1.1) \quad \check{C}(X, Y)Z \\
= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\
- \frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)[g(Y, Z)X - g(X, Z)Y],$$

where a and b are constants and R, S, Q and r are the Riemannian curvature-tensor, the Ricci - tensor, the Ricci operator and the scalar curvature of the manifold respectively. A $(2n + 1)$ -dimensional Riemannian manifold (M, g)

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is quasi-Conformally flat if $\check{C} = 0$. Various properties of the quasi-conformal curvature tensor on contact metric manifolds have been studied by several geometers [1, 4, 5, 13] etc. If $a = 1$ and $b = -\frac{1}{2n-1}$, then $\check{C}$ becomes the conformal curvature tensor C which is given by

$$(1.2) \quad C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} \{g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y\} \\ + \frac{r}{2n(2n-1)} \{g(Y, Z)X - g(X, Z)Y\}.$$

The Weyl projective curvature tensor P of type $(1, 3)$ on a $(2n+1)$ -dimensional Riemannian manifold (M, g) is defined as

$$(1.3) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{2n} [S(Y, Z)X - S(X, Z)Y],$$

for any $X, Y, Z \in TM$. The manifold (M, g) is said to be projectively flat if P vanishes identically on M . In this paper we study quasi-conformally flat and projectively flat trans-Sasakian manifolds.

2. PRELIMINARIES

Let M be a $(2n+1)$ -dimensional almost contact metric manifold [1] with almost contact metric structure (ϕ, ξ, η, g) , where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric on M such that

$$(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0,$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.3) \quad g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X),$$

for all $X, Y \in TM$.

An almost contact metric manifold is said to be contact manifold if

$$(2.4) \quad d\eta(X, Y) = \Phi(X, Y) = g(X, \phi Y),$$

$\Phi(X, Y)$ is being called fundamental 2-form of M .

An almost contact metric manifold M is called trans-Sasakian manifold if

$$(2.5) \quad (\nabla_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta g\{(\phi X, Y)\xi - \eta(Y)\phi X\},$$

where ∇ is Levi-Civita connection of Riemannian metric g and α and β are smooth functions on M . From equation (2.5) and equations (2.1), (2.2) and (2.3), we get

$$(2.6) \quad (\nabla_X \phi)\xi = -\alpha\phi X + \beta[X - \eta(X)\xi],$$

$$(2.7) \quad (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

In a $(2n + 1)$ - dimensional trans-Sasakian manifold, we have ([6] , [3])

$$(2.8) \quad \begin{aligned} R(X, Y) \xi = & (\alpha^2 - \beta^2) [\eta(Y) X - \eta(X) Y] \\ & + 2\alpha\beta [\eta(Y) \phi X - \eta(X) \phi Y] \\ & + (Y\alpha) \phi X - (X\alpha) \phi Y \\ & + (Y\beta) \phi^2 X - (X\beta) \phi^2 Y, \end{aligned}$$

$$(2.9) \quad \begin{aligned} R(\xi, Y) X = & (\alpha^2 - \beta^2) [g(X, Y) \xi - \eta(X) Y] \\ & + 2\alpha\beta [g(\phi X, Y) \xi - \eta(X) \phi Y] \\ & + (X\alpha) \phi Y + g(\phi X, Y) (\text{grad} \alpha) \\ & + (X\beta) [Y - \eta(Y) \xi] - g(\phi X, \phi Y) (\text{grad} \beta), \end{aligned}$$

$$(2.10) \quad \begin{aligned} \eta(R(\xi, Y) X) \\ = g(R(\xi, Y) X, \xi) = (\alpha^2 - \beta^2 - \xi\beta) [g(X, Y) - \eta(X) \eta(Y)] \end{aligned}$$

and

$$(2.11) \quad 2\alpha\beta + \xi\alpha = 0.$$

In a trans-Sasakian manifold, we also have [6]

$$(2.12) \quad S(X, \xi) = (2n(\alpha^2 - \beta^2) - \xi\beta) \eta(X) - (2n - 1) X\beta - (\phi X) \alpha$$

and

$$(2.13) \quad Q\xi = (2n(\alpha^2 - \beta^2) - \xi\beta) \xi - (2n - 1) \text{grad} \beta + \phi(\text{grad} \alpha).$$

An almost contact metric manifold M is said to be η -Einstein if its Ricci-tensor S is of the form

$$(2.14) \quad S(X, Y) = ag(X, Y) + b\eta(X) \eta(Y),$$

where a and b are smooth functions on M . An η -Einstein manifold becomes Einstein manifold if $b = 0$, i.e

$$(2.15) \quad S(X, Y) = ag(X, Y).$$

If $\{e_1, e_2, \dots, e_{2n}, e_{2n+1} = \xi\}$ be a local ortho-normal basis of tangent space in a $(2n + 1)$ - dimensional almost contact manifold M , then we have

$$(2.16) \quad \sum_{i=1}^{2n+1} g(e_i, e_i) = (2n + 1).$$

$$(2.17) \quad \sum_{i=1}^{2n+1} g(e_i, Y) S(X, e_i) = \sum_{i=1}^{2n+1} R(e_i, Y, X, e_i) = S(X, Y).$$

3. QUASI-CONFORMALLY FLAT AND PROJECTIVELY FLAT MANIFOLDS

Let M be a $(2n + 1)$ - dimensional quasi-conformally flat manifold, then from equation (1.1) we have

$$(3.1) \quad R(X, Y)Z = \frac{b}{a}[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY - g(Y, Z)QX] \\ + \frac{r}{(2n + 1)a}(\frac{a}{2n} + 2b)[g(Y, Z)X - g(X, Z)Y].$$

Let $\{e_1, e_2, \dots, e_{2n}, e_{2n+1} = \xi\}$ be a local ortho-normal basis of tangent space. Putting $Y = Z = e_i$, in equation (3.1), we get

$$(3.2) \quad S(X, W) = \frac{r}{2n + 1}g(X, W) \quad \text{if } a + (2n - 1)b \neq 0.$$

Hence from equation (3.2), we can state the following theorem:

Theorem 3.1. *A quasi-conformally flat manifold is an Einstein manifold if $a + (2n - 1)b \neq 0$. If $a + (2n - 1)b = 0$, then $\check{C}(X, Y)Z = aC(X, Y)Z$, or $\check{C}(X, Y)Z = -(2n - 1)bC(X, Y)Z$.*

This leads to:

Corollary 3.2. *A quasi-conformally flat manifold is conformally flat if $a + (2n - 1)b = 0$ and $a \neq 0$ (equivalently if $a + (2n - 1)b = 0$ and $b \neq 0$).*

Corollary 3.3. *If $a + (2n - 1)b = 0$, then quasi-conformally curvature becomes constant multiple of conformal curvature tensor.*

If the manifold is projectively flat then from equation(1.3), we have

$$(3.3) \quad R(X, Y)Z = \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y].$$

Putting $Y = Z = e_i$ in equation (3.3), we get

$$(3.4) \quad S(X, W) = \frac{1}{2n}[rg(X, W) - S(X, W)],$$

$$(3.5) \quad S(X, W) = \frac{r}{2n + 1}g(X, W).$$

Hence a projectively flat manifold is an Einstein manifold.

4. QUASI-CONFORMALLY FLAT TRANS-SASAKIAN MANIFOLDS

If a trans-Sasakian manifold is quasi-conformally flat then from equation (1.1) we have

$$(4.1) \quad R(X, Y, Z, W) = \frac{b}{a}[S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + g(X, Z)S(Y, W) - g(Y, Z)S(X, W)] + \frac{r}{(2n+1)a}(\frac{a}{2n} + 2b)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

On putting $Z = \xi$ in equation (4.1) and using equations (2.3) and (2.12), we get

$$(4.2) \quad g(R(X, Y)\xi, W) = \frac{b}{a}[\{(2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\phi X)\alpha\}g(Y, W) - \{(2n(\alpha^2 - \beta^2) - \xi\beta)\eta(Y) - (2n-1)Y\beta - (\phi Y)\alpha\}g(X, W) + \eta(X)S(Y, W) - \eta(Y)S(X, W)] + \frac{r}{(2n+1)a}(\frac{a}{2n} + 2b)\{\eta(Y)g(X, W) - \eta(X)g(Y, W)\}$$

Again putting $X = \xi$ in equation (4.2) and using equations (2.1), (2.3), (2.10), (2.11) and (2.12), we get

$$(4.3) \quad S(Y, W) = [\frac{r}{(2n+1)a}(\frac{a}{2n} + 2b) - \frac{a}{b}(2n(\alpha^2 - \beta^2) - \xi\beta)]g(Y, W) + [2(2n(\alpha^2 - \beta^2) - \xi\beta) + \frac{a}{b}(\alpha^2 - \beta^2 - \xi\beta) - \frac{r}{(2n+1)b}(\frac{a}{2n} + 2b)]\eta(Y)\eta(W) - \{(2n-1)Y\beta - (\phi Y)\alpha\}\eta(W) - \{((2n-1)W\beta - (\phi W)\alpha)\}\eta(Y).$$

From equations (3.2) and (4.3), we get

$$(4.4) \quad r = 2n(2n+1)(\alpha^2 - \beta^2 - \xi\beta),$$

and

$$(4.5) \quad (2n-1)\text{grad } \beta - \phi(\text{grad } \alpha) = (2n-1)(\xi\beta)\xi.$$

This leads to:

Theorem 4.1. *A trans-Sasakian manifold can not be quasi-conformally flat unless $(2n-1)\text{grad } \beta - \phi(\text{grad } \alpha) - (2n-1)(\xi\beta)\xi$ is zero.*

From the equations (4.4) and (4.5), we have the following:

Corollary 4.2. *If a trans-Sasakian manifold is quasi-conformally flat then scalar curvature tensor $r = 2n(2n+1)(\alpha^2 - \beta^2 - \xi\beta)$, where α and β are related by $(2n-1)\text{grad}\beta - \phi(\text{grad}\alpha) = (2n-1)(\xi\beta)\xi$.*

5. PROJECTIVELY FLAT TRANS-SASAKIAN MANIFOLDS

If a trans-Sasakian manifold is projectively flat then from equation (1.3), we have

$$(5.1) \quad R(X, Y, Z, W) = g(R(X, Y)Z, W) \\ = \frac{1}{2n}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)].$$

On putting $W = \xi$ in equation (5.1) and using equations (2.3) and (2.12), we get

$$(5.2) \quad \eta((R(X, Y)Z)) = \frac{1}{2n}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)].$$

Again putting $X = \xi$ in equation (5.2) and using equations (2.1), (2.3), (2.10), (2.11) and (2.12), we get

$$S(Y, Z) = 2n(\alpha^2 - \beta^2 - \xi\beta)g(Y, Z) + (2n-1)(\xi\beta)\eta(Y)\eta(Z) \\ + ((2n-1)(Z\beta) + (\phi Z)\alpha)\eta(Y).$$

From equation (3.5) and (5.3), we get

$$(5.3) \quad r = 2n(2n+1)(\alpha^2 - \beta^2 - \xi\beta),$$

and

$$(5.4) \quad (2n-1)(d\beta - \xi(\beta)\eta) + d\alpha o\phi = 0.$$

This leads to:

Theorem 5.1. *A trans-Sasakian manifold can not be projectively flat unless $(2n-1)(d\beta - (\xi\beta)\eta) + d\alpha o\phi$ is zero.*

From the equations (5.4) and (5.5), we have the following:

Corollary 5.2. *If a trans-Sasakian manifold is projectively flat then scalar curvature tensor $r = 2n(2n+1)(\alpha^2 - \beta^2 - \xi\beta)$, where α and β are related $(2n-1)(d\beta - \xi(\beta)\eta) + d\alpha o\phi = 0$.*

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