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# EINSTEIN METRICS ON WARPED PRODUCT FINSLER SPACES

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Abstract. In this paper, we prove that a warped product Finsler metric is an Einstein metric if and only if some partial differential equations are satisfied. Several results are obtained in special cases, for example, the case of Riemannian, Locally Minkowski, and Berwald spaces. Moreover, we present the static vacuum Einstein equations on Finsler manifold.

## 1. Introduction

The warped product is an important concept in geometry and physics. This concept was first introduced by Bishop and O'Neill to construct Riemannian manifolds with negative curvature [11]. It has been applied for the construction of Einstein metrics on noncompact complete Riemannian manifolds and other important examples in relativity and differential geometry [8],[11]. Besse produced a non-trivial Einstein warped product on a compact Riemannian manifold [10]. S. Kim established compact base manifolds with a positive scalar curvature which do not admit any non-trivial Einstein warped product [15].

One of the important problems in Finsler geometry is to characterize and construct the Einstein metrics, constant Ricci curvature metrics and, as a special case, constant flag curvature metrics. Many valuable results have been achieved, most of which are related to a special class of Finsler metrics named $(\alpha, \beta)$-metrics due to its computability. In 2004, D. Bao, C. Robles and Shen classified Randers metrics with constant flag curvature [7]. Also with the help of the navigation problem, D. Bao and C. Robles give a characterization for Einstein metrics of Randers type [6].

Warped product extended for Finslerian metrics by the work of Kozma et al [16]. Some objects of Riemannian manifolds are expanded to the warped product Finsler manifolds in [4], [5]. Two authors of this paper presented some necessary and sufficient conditions which the spray manifold is projectively equivalent to the warped product Finsler manifolds [18]. Also, they found

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the necessary and sufficient conditions for the Sasaki-Finsler metric of the warped product Finsler manifold to be bundle like for the vertical foliation [2]. In [19] and [20] , Tayebi and his collaborates studied the warped and doubly warped product structure in Finsler geometry. They consider warped product Finsler metrics with scalar flag curvature and some well-known non-Riemannian curvature properties such as Berwald, Landsberg, and relatively isotropic (mean) Landsberg curvatures. Inspired by the mentioned works, we consider Einstein warped product Finsler metrics and present necessary and sufficient conditions for the warped product Finsler space to be Einstein . Moreover, several results are obtained in the special cases, for example, the case of Riemannian, Locally Minkowski and Berwald spaces are considered. Also, we present the static vacuum Einstein equations on Finsler manifold.

## 2. Preliminaries

Let $\mathbb{F}_1 = (M_1, F_1)$ and $\mathbb{F}_2 = (M_2, F_2)$ be two Finsler manifolds and $f\colon M_1 \to \mathbb{R}$ be a non-negative smooth function. Consider $\mathbb{F} = (M, F)$ where $M$ is the product manifold $M_1 \times M_2$ and the function $F$ is defined as

$$F^2(x_1, x_2, y_1, y_2) = F_1^2(x_1, y_1) + f(x_1)^2 F_2^2(x_2, y_2). \tag{1}$$

The function $F$ is smooth on $TM^0$ and is obviously positively homogeneous of degree 1 with respect to $(y_1, y_2)$. For the function $F$,

$$(g_{ab}(x,y)) = (\frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b}) = \begin{pmatrix} \hat{g}_{ij}(x_1, y_1) & 0 \\ 0 & f^2(x_1)\check{g}_{\alpha\beta}(x_2, y_2) \end{pmatrix} \tag{2}$$

are the components of a positive definite quadratic form at every point $(x, y)$. Therefore $\mathbb{F} = (M, F)$ defines a Finsler manifold and is called a warped product of $\mathbb{F}_1$ and $\mathbb{F}_2$. The warped product Finsler metric $F$ is denoted by $F = F_1 \times_f F_2$ and the function $f$ is called warping function [2],[16].

*Notation.* Lowercase Latin letters like $\{i, j, k, l, \ldots\}$ , $\{\alpha, \beta, \gamma, \ldots\}$ and $\{a, b, c, d, \ldots\}$ are used in the upper position for variable indices. They belong to the set $\{1, \ldots, m_1\}$, $\{1, \ldots, m_2\}$ and $\{1, \ldots, m_1 + m_2\}$ respectively, according to the spaces $\mathbb{F}_1$, $\mathbb{F}_2$ or $\mathbb{F} = \mathbb{F}_1 \times_f \mathbb{F}_2$ they represent. Variables of $\mathbb{F}_1$ and $\mathbb{F}_2$ have lower indices 1 and 2 respectively, like $x_1^i, y_1^j$ and $x_2^\alpha, y_2^\beta$.

When there is no appropriate position to place indices 1 and 2, objects of $\mathbb{F}_1$ and $\mathbb{F}_2$ will be hat and check respectively, like $\hat{g}_{ij}$ and $\check{g}_{\alpha\beta}$, to indicate their relevant spaces.

The inverse $g^{ab}$ of $g_{ab}$ is given by

$$(g^{ab}(\mathtt{x}, \mathtt{y})) = \begin{pmatrix} \hat{g}^{ij}(x_1, y_1) & 0 \\ 0 & f^{-2}(x_1)\check{g}^{\alpha\beta}(x_2, y_2) \end{pmatrix}. \tag{3}$$

The local coordinates $(x_1, x_2, y_1, y_2)$ on $TM^0$ are transformed by the rules

$$\tilde{x}_1^i = \tilde{x}_1^i(x_1^1, \ldots, x_1^{m_1}), \quad \tilde{y}_1^i = \frac{\partial \tilde{x}_1^i}{\partial x_1^j} y_1^j,$$
$$\tilde{x}_2^\alpha = \tilde{x}_2^\alpha(x_2^1, \ldots, x_2^{m_2}), \quad \tilde{y}_2^\alpha = \frac{\partial \tilde{x}_2^\alpha}{\partial x_2^\beta} y_2^\beta.$$

And for $\frac{\partial}{\partial y^a}$, we have

$$\frac{\partial}{\partial y_1^i} = \frac{\partial \tilde{x}_1^j}{\partial x_1^i} \frac{\partial}{\partial \tilde{y}_1^j}, \quad \frac{\partial}{\partial y_2^\alpha} = \frac{\partial \tilde{x}_2^j}{\partial x_2^i} \frac{\partial}{\partial \tilde{y}_2^j}. \tag{4}$$

So, the vertical distribution $\mathcal{V}TM^0$ is spanned by $\{\frac{\partial}{\partial y_1^i}, \frac{\partial}{\partial y_2^\alpha}\}$ and the horizontal distribution $\mathcal{H}TM^0$ is spanned by $\{\frac{\delta}{\delta x_1^i}, \frac{\delta}{\delta x_2^\alpha}\}$ which

$$\begin{aligned} \tfrac{\delta}{\delta x_1^i} &= \tfrac{\partial}{\partial x_1^i} - G_i^j \tfrac{\partial}{\partial y_1^j} - G_i^\beta \tfrac{\partial}{\partial y_2^\beta}, \\ \tfrac{\delta}{\delta x_2^\alpha} &= \tfrac{\partial}{\partial x_2^\alpha} - G_\alpha^j \tfrac{\partial}{\partial y_1^j} - G_\alpha^\beta \tfrac{\partial}{\partial y_2^\beta}. \end{aligned} \tag{5}$$

The geodesic coefficients $G^a = (G^i, G^\alpha)$ are local defined functions as

$$G^i = \hat{G}^i - \frac{1}{4} \hat{g}^{ih} \frac{\partial f^2}{\partial x_1^h} F_2^2, \quad G^\alpha = \check{G}^\alpha + \frac{1}{2} \frac{1}{f^2} y_2^\alpha y_1^h \frac{\partial f^2}{\partial x_1^h}. \tag{6}$$

Now, the coefficients $G_b^a = (G_j^i, G_\beta^i, G_j^\alpha, G_\beta^\alpha)$ of the non-linear connection are given by

$$\begin{aligned} G_j^i &= \hat{G}_j^i - \frac{1}{4} \frac{\partial \hat{g}^{ih}}{\partial y_1^j} \frac{\partial f^2}{\partial x_1^h} F_2^2, \\ G_\beta^i &= -\frac{1}{4} \hat{g}^{ih} \frac{\partial f^2}{\partial x_1^h} \frac{\partial F_2^2}{\partial y_2^\beta}, \\ G_j^\alpha &= \frac{1}{2} \frac{1}{f^2} y_2^\alpha \frac{\partial f^2}{\partial x_1^j}, \\ G_\beta^\alpha &= \check{G}_\beta^\alpha + \frac{1}{2} \frac{1}{f^2} y_1^h \frac{\partial f^2}{\partial x_1^h} \delta_\beta^\alpha. \end{aligned} \tag{7}$$

**Corollary 2.1.** *The coefficients* $G^a_{bc} = (G^i_{jk}, G^i_{\beta k}, G^i_{\beta\gamma}, G^\alpha_{jk}, G^\alpha_{j\gamma}, G^\alpha_{\beta\gamma})$ *of the nonlinear connection on the warped product Finsler manifold are gotten as*

$$
\begin{aligned}
G^i_{jk} &= \hat{G}^i_{jk} - \frac{1}{4}\frac{\partial^2 \hat{g}^{ih}}{\partial y_1^j \partial y_1^k}\frac{\partial f^2}{\partial x_1^h}F_2^2 = G^i_{kj},\\
G^i_{\beta k} &= -\frac{1}{4}\frac{\partial \hat{g}^{ih}}{\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\frac{\partial F_2^2}{\partial y_2^\beta} = G^i_{k\beta},\\
G^i_{\beta\gamma} &= -\frac{1}{2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\check{g}_{\beta\gamma} = G^i_{\gamma\beta}\ ,\\
G^\alpha_{jk} &= 0,\\
G^\alpha_{j\gamma} &= \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x_1^j}\delta^\alpha_\gamma = G^\alpha_{\gamma j},\\
G^\alpha_{\beta\gamma} &= \check{G}^\alpha_{\beta\gamma} = G^\alpha_{\gamma\beta}.
\end{aligned}
\tag{8}
$$

*Proof.* Using $G^a_{bc} = \frac{\partial G^a_b}{\partial y^c}$, for detailed see [2]. □

## 3. The Laplacian of the Sasaki Finsler metrics

It is well known that various kinds of Laplace operators play a very important role in differential geometry and physics, especially in the theory of harmonic integral and Bochner technique. In [14], Chunping and Tongde generalized the Laplace operator in Riemannian manifolds to Finsler vector bundles as such bundles arise naturally in Finsler geometry. Using the $h$-Laplace operator, they proved some integral formulas for horizontal Finsler vector fields and scalar fields on vector bundles.

Let $(M, F)$ be a Finsler manifold. Then the tangent bundle $TM$ endowed with the Sasaki-type metric constructed from the given Finsler metric $F$ is a Riemannian vector bundle. Consider

$$
\mathrm{d}V = \det(g_{ij})\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \wedge \delta y^1 \wedge \ldots \wedge \delta y^m \tag{9}
$$

be the volume form associated with the Riemannian structure, $G = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}\delta y^i \otimes \delta y^j$, on $TM$ and $\mathcal{L}_X$ be the Lie derivative with respect to $X \in \mathcal{X}(TM)$, then the notations as gradient and divergent on $TM$ can be introduced. The divergence of $X = X^i\frac{\delta}{\delta x^i} + \bar{X}^i\frac{\partial}{\partial y^i}$ is defined by

$$
\mathcal{L}_X \mathrm{d}V = (\operatorname{div} X)\mathrm{d}V. \tag{10}
$$

We denote $\operatorname{div}_h X =: \operatorname{div}(X^i\frac{\delta}{\delta x^i})$ and $\operatorname{div}_v X =: \operatorname{div}(\bar{X}^i\frac{\partial}{\partial y^i})$, then we have the following Lemma.

**Lemma 3.1.** *Let* $X = X^i\frac{\delta}{\delta x^i} + \bar{X}^i\frac{\partial}{\partial y^i} \in \mathcal{X}(TM)$ *then*

$$
\operatorname{div} X = \underset{h}{\operatorname{div}} X + \underset{v}{\operatorname{div}} X, \tag{11}
$$

*where*

$$\operatorname*{div}_h X = \nabla_{\frac{\delta}{\delta x^i}} X^i - P^k_{ik} X^i, \quad \operatorname*{div}_v X = \nabla_{\frac{\partial}{\partial y^i}} \bar{X}^i + C^k_{ik} \bar{X}^i,$$

*which* $P^k_{ij} = G^k_{ij} - F^k_{ij}$.

*Proof.* Applying proposition (3.1) of [14] on Finsler spaces. □

Now if we define $\operatorname{grad}(f) = \nabla f$ by

$$G(\nabla f, X) = Xf, \quad \forall X \in \mathcal{X}(TM), \quad f \in C^\infty(TM)$$

then in the adapted frame $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, we have

$$\nabla f = \nabla_h f + \nabla_v f, \tag{12}$$

where

$$\nabla_h f = g^{ij} \frac{\delta f}{\delta x^j} \frac{\delta}{\delta x^i}, \quad \nabla_v f = g^{ij} \frac{\partial f}{\partial y^j} \frac{\partial}{\partial y^i}.$$

If $f \in C^\infty(M)$ then we have

$$G(\nabla_h f^2, \nabla_h f^2) = g^{ij} \frac{\partial f^2}{\partial x^i} \frac{\partial f^2}{\partial x^j}. \tag{13}$$

Now the $h(v)$-Laplace operator on $TM$ is defined by

$$\Delta_h := \operatorname*{div}_h \circ \nabla_h, \quad \Delta_v := \operatorname*{div}_v \circ \nabla_v. \tag{14}$$

**Lemma 3.2.** *Let $(M, F)$ be Finsler space and $f \in C^\infty(TM)$. Then*

$$\Delta f = \Delta_h f + \Delta_v f, \tag{15}$$

*where*

$$\Delta_h f = \frac{\delta g^{ij}}{\delta x^i} \frac{\delta f}{\delta x^j} + g^{ij} \frac{\delta}{\delta x^i}(\frac{\delta f}{\delta x^j}) - P^k_{ik} g^{ij} \frac{\delta f}{\delta x^j},$$
$$\Delta_v f = \frac{\partial g^{ij}}{\partial y^i} \frac{\partial f}{\partial y^j} + g^{ij} \frac{\partial}{\partial y^i}(\frac{\partial f}{\partial y^j}) + g^{ij} \frac{\partial f}{\partial y^j} C^k_{ik}.$$

*Proof.* See [14]. □

Consider the Finsler manifold to be Riemannian space and $f \colon M \to \mathbb{R}$, so the horizontal Laplacian of $f^2$ is given by

$$\Delta_h f^2 = \frac{\partial g^{ij}}{\partial x^i} \frac{\partial f^2}{\partial x^j} + g^{ij} \frac{\partial^2 f^2}{\partial x^i \partial x^j} + G^k_{ik} g^{ij} \frac{\partial f^2}{\partial x^j}. \tag{16}$$

When the Finsler space is locally Minkowski space, the horizontal Laplacian of $f \in C^\infty(M)$ is gotten by

$$\Delta_h f^2 = g^{ij} \frac{\partial^2 f^2}{\partial x^i \partial x^j}. \tag{17}$$

## 4. Einstein Metrics

The importance of the Ricci tensor can be seen from the Bonnet-Myers theorem. The Riemannian version of this result is one of the most useful comparison theorems in differential geometry [13]. It was first extended to the Finsler manifolds by the work of Auslander [3]. Akbar-Zadeh generalized the concept of Ricci tensor on Finsler geometry [1]. In this section, we perform the concept of Ricci tensor on warped product Finsler manifolds and present some conditions for the warped product Finsler metric to be Einstein.

Let us begin by introducing *Ricci scalar* for the warped product Finsler space $\mathbb{F} = (M, F)$ as follows

$$\mathfrak{Ric} := R^a_a = R^i_i + R^\alpha_\alpha, \tag{18}$$

where

$$\begin{aligned} R^i_i &:= \frac{1}{F^2}(R^i_{jik}y^j_1 y^k_1 + R^i_{\beta i\gamma}y^\beta_2 y^\gamma_2), \\ R^\alpha_\alpha &:= \frac{1}{F^2}(R^\alpha_{j\alpha k}y^j_1 y^k_1 + R^\alpha_{\beta\alpha\gamma}y^\beta_2 y^\gamma_2), \end{aligned}$$

and $R^a_{bcd}$ are the $h$-curvature tensor field of the Cartan connection of $\mathbb{F}$. We define *Ricci tensor* from the Ricci scalar as follows

$$\mathrm{Ric}_{bc} := \frac{\partial^2(\frac{1}{2}F^2\mathfrak{Ric})}{\partial y^b \partial y^c}. \tag{19}$$

The definition of the Ricci tensor is not practical if one wants to compute it. So we use the generalized Berwald's formula on warped product Finsler manifold that is defined as

$$K^a_a := 2\tfrac{\partial G^a}{\partial x^a} - \tfrac{\partial G^a}{\partial y^b}\tfrac{\partial G^b}{\partial y^a} - y^b\tfrac{\partial G^a}{\partial x^b \partial y^a} + 2G^b\tfrac{\partial G^a}{\partial y^b \partial y^a}. \tag{20}$$

The Ricci scalar is related to the generalized Berwald's Formula in the following manner:

$$F^2 R^a_a = K^a_a. \tag{21}$$

Then the Ricci tensor of the warped product Finsler space is given by

$$\begin{aligned}
\mathrm{Ric}_{kl} = \hat{\mathrm{Ric}}_{kl} &+ \frac{1}{2}F_2^2[-\frac{1}{2}\frac{\partial^3\hat{g}^{ih}}{\partial x_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h}+\frac{1}{4}y_1^j\frac{\partial^4\hat{g}^{ih}}{\partial x_1^j\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h}\\
&-\frac{1}{2}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^k\partial y_1^l}\frac{\partial^2 f^2}{\partial x_1^h\partial x_1^i}+\frac{1}{4}y_1^j\frac{\partial^3\hat{g}^{ih}}{\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial^2 f^2}{\partial x_1^j\partial x_1^h}+\frac{1}{4}\frac{1}{f^2}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i}\\
&-\frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^3\hat{g}^{is}}{\partial y_1^i\partial y_1^k\partial y_1^l}]-\frac{1}{2}\frac{\partial^2\ln f^2}{\partial x_1^k\partial x_1^l}+\frac{1}{2}\frac{1}{f^2}\hat{G}^j_{kl}\frac{\partial f^2}{\partial x_1^j}\\
&+\frac{1}{8}\frac{\partial f^2}{\partial x_1^s}F_2^2[\hat{G}^i_{klj}\frac{\partial\hat{g}^{js}}{\partial y_1^i}+\hat{G}^i_{kj}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^l}+\hat{G}^i_{lj}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k}+\hat{G}^i_j\frac{\partial^3\hat{g}^{js}}{\partial y_1^i\partial y_1^k\partial y_1^l}]\\
&+\frac{1}{8}\frac{\partial f^2}{\partial x_1^h}F_2^2[\hat{G}^j_{kli}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}+\hat{G}^j_{ki}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^l}+\hat{G}^j_{li}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}+\hat{G}^j_i\frac{\partial^3\hat{g}^{ih}}{\partial y_1^j\partial y_1^k\partial y_1^l}]\\
&-\frac{1}{4}\frac{\partial f^2}{\partial x_1^s}F_2^2[\hat{G}^j_{kl}\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i}+\hat{G}^j_k\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^l}+\hat{G}^j_l\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}\\
&+\hat{G}^j\frac{\partial^4\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k\partial y_1^l}]+\frac{1}{8}y_2^\beta\frac{\partial^3\hat{g}^{ih}}{\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h}\frac{\partial F_2^2}{\partial x_2^\beta}-\frac{1}{4}\check{G}^\beta\frac{\partial^3\hat{g}^{is}}{\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^s}\frac{\partial F_2^2}{\partial y_2^\beta}\\
&-\frac{1}{4}\frac{\partial f^2}{\partial x_1^h}F_2^2[\hat{G}^i_{ji}\frac{\partial^2\hat{g}^{jh}}{\partial y_1^k\partial y_1^l}+\hat{G}^i_{lji}\frac{\partial\hat{g}^{jh}}{\partial y_1^k}+\hat{G}^i_{kji}\frac{\partial\hat{g}^{jh}}{\partial y_1^l}+\hat{G}^i_{klji}\hat{g}^{jh}]\\
&+\frac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}F_2^4[-\frac{1}{16}\frac{\partial^3\hat{g}^{ih}}{\partial y_1^j\partial y_1^k\partial y_1^l}\frac{\partial\hat{g}^{js}}{\partial y_1^i}-\frac{1}{16}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^l}\\
&-\frac{1}{16}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^l}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k}-\frac{1}{16}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial^3\hat{g}^{js}}{\partial y_1^i\partial y_1^k\partial y_1^l}+\frac{1}{8}\frac{\partial^2\hat{g}^{jh}}{\partial y_1^k\partial y_1^l}\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i}\\
&+\frac{1}{8}\frac{\partial\hat{g}^{jh}}{\partial y_1^k}\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^l}+\frac{1}{8}\frac{\partial\hat{g}^{jh}}{\partial y_1^l}\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}+\frac{1}{8}\hat{g}^{jh}\frac{\partial^4\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k\partial y_1^l}]
\end{aligned}\tag{22}$$

$$\begin{aligned}
\mathrm{Ric}_{k\nu} = &\frac{1}{2}\frac{\partial F_2^2}{\partial y_2^\nu}[-\frac{1}{2}\frac{\partial^2\hat{g}^{ih}}{\partial x_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}-\frac{1}{2}\frac{\partial\hat{g}^{ih}}{\partial y_1^k}\frac{\partial^2 f^2}{\partial x_1^h\partial x_1^i}+\frac{1}{4}\hat{G}^i_{kj}\frac{\partial\hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}\\
&+\frac{1}{4}\hat{G}^i_j\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^s}+\frac{1}{4}\hat{G}^j_{ki}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}+\frac{1}{4}\hat{G}^j_i\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\\
&+\frac{1}{4}y_1^j\frac{\partial^3\hat{g}^{ih}}{\partial x_1^j\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}+\frac{1}{4}y_1^j\frac{\partial^2\hat{g}^{ih}}{\partial y_1^i\partial y_1^k}\frac{\partial^2 f^2}{\partial x_1^j\partial x_1^h}-\frac{1}{2}\hat{G}^j_k\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}\\
&-\frac{1}{2}\hat{G}^j\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^s}-\frac{1}{2}\frac{\partial\hat{g}^{jh}}{\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji}-\frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{kji}\\
&+\frac{1}{4}\frac{1}{f^2}\frac{\partial\hat{g}^{ih}}{\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i}-\frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^2\hat{g}^{is}}{\partial y_1^i\partial y_1^k}]+\frac{1}{8}y_2^\beta\frac{\partial^2\hat{g}^{ih}}{\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\frac{\partial^2 F_2^2}{\partial x_2^\beta\partial y_2^\nu}\\
&+\frac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial F_2^4}{\partial y_2^\nu}[-\frac{1}{16}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}\frac{\partial\hat{g}^{js}}{\partial y_1^i}-\frac{1}{16}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k}+\frac{1}{8}\frac{\partial\hat{g}^{jh}}{\partial y_1^k}\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i}
\end{aligned}\tag{23}$$

$$+\frac{1}{8}\hat{g}^{jh}\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}]-\frac{1}{4}\frac{\partial^2\hat{g}^{is}}{\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^s}[\breve{G}^\beta_\nu\frac{\partial F_2^2}{\partial y_2^\beta}+\breve{G}^\beta\frac{\partial^2 F_2^2}{\partial y_2^\beta\partial y_2^\nu}]$$

$$\begin{aligned}\mathrm{Ric}_{\mu\nu}&=\breve{\mathrm{Ric}}_{\mu\nu}+\frac{1}{2}\frac{\partial^2 F_2^2}{\partial y_2^\mu\partial y_2^\nu}[-\frac{1}{2}\frac{\partial\hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h}-\frac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^h\partial x_1^i}+\frac{1}{4}y_1^j\frac{\partial^2\hat{g}^{ih}}{\partial x_1^j\partial y_1^i}\frac{\partial f^2}{\partial x_1^h}\\&+\frac{1}{4}y_1^j\frac{\partial\hat{g}^{ih}}{\partial y_1^i}\frac{\partial^2 f^2}{\partial x_1^j\partial x_1^h}-\frac{1}{2}\hat{G}^j\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}-\frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji}+\frac{1}{4}\frac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i}\\&-\frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial\hat{g}^{is}}{\partial y_1^i}+\frac{1}{4}\hat{G}^i_j\frac{\partial\hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}+\frac{1}{4}\hat{G}^j_i\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}]\\&-\frac{1}{4}\frac{\partial\hat{g}^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}[\breve{G}^\beta_{\mu\nu}\frac{\partial F_2^2}{\partial y_2^\beta}+\breve{G}^\beta_\mu\frac{\partial F_2^2}{\partial y_2^\beta\partial y_2^\nu}+\breve{G}^\beta\frac{\partial^3 F_2^2}{\partial y_2^\beta\partial y_2^\mu\partial y_2^\nu}+\breve{G}^\beta_\nu\frac{\partial F_2^2}{\partial y_2^\beta\partial y_2^\mu}]\\&+\frac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^2 F_2^4}{\partial y_2^\mu\partial y_2^\nu}[-\frac{1}{16}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial\hat{g}^{js}}{\partial y_1^i}+\frac{1}{8}\hat{g}^{jh}\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i}]\\&+\frac{1}{8}y_2^\beta\frac{\partial\hat{g}^{ih}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^h}\frac{\partial^3 F_2^2}{\partial x_2^\beta\partial y_2^\mu\partial y_2^\nu}+\frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\breve{G}^\alpha_{\mu\nu\alpha}\end{aligned}\tag{24}$$

**Definition 4.1.** Warped product Finsler metric $F$ is called an Einstein metric if there exists a constant $K\in\mathbb{R}$ such that

$$\mathrm{Ric}_{ab}=Kg_{ab}.\tag{25}$$

In this case, the warped product Finsler space $\mathbb{F}=(M,F)$ is called an Einstein manifold. In Riemannian geometry, the warped product manifold $(M,g)=M_1\times_f M_2$ is Einstein with $\mathrm{Ric}=Kg$ if and only if $(M_2,\breve{g})$ is Einstein, i.e., $\breve{\mathrm{Ric}}=K_2\breve{g}$ for a constant $K_2$ and the followings hold [17]:

$$\begin{aligned}K\hat{g}&=\hat{\mathrm{Ric}}-\frac{d}{f}H^f,\\K&=\frac{K_2}{f^2}-\frac{\Delta f}{f}-(d-1)|\frac{\nabla f}{f}|^2_{\hat{g}},\end{aligned}\tag{26}$$

where $\dim M_1\geq 2$, $\dim M_2=d\geq 2$ and $\Delta f=\mathrm{tr}H^f=\mathrm{tr}(\mathrm{Hess}f)$.

Now, we want to generalize this result on warped product Finsler manifolds. In fact, we answer Chern's question on warped product Finsler space. He asked, "whenever a smooth manifold admits an Einstein Finsler metric?"

**Theorem 4.2.** *Let $\mathbb{F}=\mathbb{F}_1\times_f\mathbb{F}_2$ be a warped product Finsler space. Consider the warped product Finsler metric $F$ is an Einstein metric of constant $K$ then the following conditions hold:*

$$\hat{\mathrm{Ric}}_{kl}=K\hat{g}_{kl}-\frac{1}{2}\frac{\partial\, ln\, f^2}{\partial x_1^j}\hat{G}^j_{kl}+\frac{1}{2}\frac{\partial^2\, ln\, f^2}{\partial x_1^k\partial x_1^l},\tag{27}$$

$$
\begin{aligned}
&\hat{\Delta}_h f^2 - \frac{1}{2}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) + 2Kf^2 - 2\check{g}^{\mu\nu}\check{\mathrm{Ric}}_{\mu\nu} + P^j_{ij}\hat{g}^{ih}\frac{\partial f^2}{\partial x^h_1} = \\
&\frac{\partial f^2}{\partial x^s_1}(\frac{1}{2}\hat{G}^i_j\frac{\partial \hat{g}^{js}}{\partial y^i_1} - \hat{G}^j\frac{\partial^2 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1}) + y^h_1\check{g}^{\mu\nu}\frac{\partial \ln f^2}{\partial x^h_1}\check{G}^\alpha_{\mu\nu\alpha} \\
&-\frac{\partial f^2}{\partial x^h_1}(\frac{1}{2}\hat{G}^j_i\frac{\partial \hat{g}^{ih}}{\partial y^j_1} + \hat{g}^{jh}\hat{G}^i_{ji}).
\end{aligned}
\tag{28}
$$

*Proof.* Using (2), (23) and (25), we obtain

(29)

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial F^2_2}{\partial y^\nu_2}[-\frac{1}{2}\frac{\partial^2 \hat{g}^{ih}}{\partial x^i_1 \partial y^k_1}\frac{\partial f^2}{\partial x^h_1} - \frac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial y^k_1}\frac{\partial^2 f^2}{\partial x^h_1 \partial x^i_1} + \frac{1}{4}\hat{G}^i_{kj}\frac{\partial \hat{g}^{js}}{\partial y^i_1}\frac{\partial f^2}{\partial x^s_1} + \frac{1}{4}\hat{G}^i_j\frac{\partial^2 \hat{g}^{js}}{\partial y^i_1 \partial y^k_1}\frac{\partial f^2}{\partial x^s_1} \\
&+\frac{1}{4}\hat{G}^j_{ki}\frac{\partial \hat{g}^{ih}}{\partial y^j_1}\frac{\partial f^2}{\partial x^h_1} + \frac{1}{4}\hat{G}^j_i\frac{\partial^2 \hat{g}^{ih}}{\partial y^j_1 \partial y^k_1}\frac{\partial f^2}{\partial x^h_1} + \frac{1}{4}y^j_1\frac{\partial^3 \hat{g}^{ih}}{\partial x^j_1 \partial y^i_1 \partial y^k_1}\frac{\partial f^2}{\partial x^h_1} + \frac{1}{4}y^j_1\frac{\partial^2 \hat{g}^{ih}}{\partial y^i_1 \partial y^k_1}\frac{\partial^2 f^2}{\partial x^j_1 \partial x^h_1} \\
&-\frac{1}{2}\hat{G}^j_k\frac{\partial^2 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1}\frac{\partial f^2}{\partial x^s_1} - \frac{1}{2}\hat{G}^j\frac{\partial^3 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1 \partial y^k_1}\frac{\partial f^2}{\partial x^s_1} - \frac{1}{2}\frac{\partial \hat{g}^{jh}}{\partial y^k_1}\frac{\partial f^2}{\partial x^h_1}\hat{G}^i_{ji} - \frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x^h_1}\hat{G}^i_{kji} \\
&+\frac{1}{4}\frac{1}{f^2}\frac{\partial \hat{g}^{ih}}{\partial y^k_1}\frac{\partial f^2}{\partial x^h_1}\frac{\partial f^2}{\partial x^i_1} - \frac{1}{2}\frac{1}{f^2}y^h_1\frac{\partial f^2}{\partial x^h_1}\frac{\partial f^2}{\partial x^s_1}\frac{\partial^2 \hat{g}^{is}}{\partial y^i_1 \partial y^k_1}] = -\frac{1}{8}y^\beta_2\frac{\partial^2 \hat{g}^{ih}}{\partial y^i_1 \partial y^k_1}\frac{\partial f^2}{\partial x^h_1}\frac{\partial^2 F^2_2}{\partial x^\beta_2 \partial y^\nu_2} \\
&-\frac{1}{2}\frac{\partial f^2}{\partial x^h_1}\frac{\partial f^2}{\partial x^s_1}\frac{\partial F^4_2}{\partial y^\nu_2}[-\frac{1}{16}\frac{\partial^2 \hat{g}^{ih}}{\partial y^j_1 \partial y^k_1}\frac{\partial \hat{g}^{js}}{\partial y^i_1} - \frac{1}{16}\frac{\partial \hat{g}^{ih}}{\partial y^j_1}\frac{\partial^2 \hat{g}^{js}}{\partial y^i_1 \partial y^k_1} + \frac{1}{8}\frac{\partial \hat{g}^{jh}}{\partial y^k_1}\frac{\partial^2 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1} \\
&+\frac{1}{8}\hat{g}^{jh}\frac{\partial^3 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1 \partial y^k_1}] + \frac{1}{4}\frac{\partial^2 \hat{g}^{is}}{\partial y^i_1 \partial y^k_1}\frac{\partial f^2}{\partial x^s_1}[\check{G}^\beta_\nu\frac{\partial F^2_2}{\partial y^\beta_2} + \check{G}^\beta\frac{\partial^2 F^2_2}{\partial y^\beta_2 \partial y^\nu_2}]
\end{aligned}
$$

Differentiating (29) with respect to $y^l_1$ and then contracting it with $\frac{1}{2}y^\nu_2$. By inserting this result into (22), Eq. (25) can be written as

$$
\begin{aligned}
&\hat{\mathrm{Ric}}_{kl} - K\hat{g}_{kl} + \frac{1}{4}\check{G}^\beta\frac{\partial^3 \hat{g}^{is}}{\partial y^i_1 \partial y^k_1 \partial y^l_1}\frac{\partial f^2}{\partial x^s_1}\frac{\partial F^2_2}{\partial y^\beta_2} + \frac{1}{2}\frac{1}{f^2}\hat{G}^j_{kl}\frac{\partial f^2}{\partial x^j_1} - \frac{1}{2}\frac{\partial^2\, ln\, f^2}{\partial x^k_1 \partial x^l_1} = \\
&\frac{1}{2}\frac{\partial f^2}{\partial x^h_1}\frac{\partial f^2}{\partial x^s_1}F^4_2[-\frac{1}{16}\frac{\partial^3 \hat{g}^{ih}}{\partial y^j_1 \partial y^k_1 \partial y^l_1}\frac{\partial \hat{g}^{js}}{\partial y^i_1} \\
&-\frac{1}{16}\frac{\partial^2 \hat{g}^{ih}}{\partial y^j_1 \partial y^k_1}\frac{\partial^2 \hat{g}^{js}}{\partial y^i_1 \partial y^l_1} - \frac{1}{16}\frac{\partial^2 \hat{g}^{ih}}{\partial y^j_1 \partial y^l_1}\frac{\partial^2 \hat{g}^{js}}{\partial y^i_1 \partial y^k_1} \\
&-\frac{1}{16}\frac{\partial \hat{g}^{ih}}{\partial y^j_1}\frac{\partial^3 \hat{g}^{js}}{\partial y^i_1 \partial y^k_1 \partial y^l_1} + \frac{1}{8}\frac{\partial^2 \hat{g}^{jh}}{\partial y^k_1 \partial y^l_1}\frac{\partial^2 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1} + \frac{1}{8}\frac{\partial \hat{g}^{jh}}{\partial y^k_1}\frac{\partial^3 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1 \partial y^l_1} \\
&+\frac{1}{8}\frac{\partial \hat{g}^{jh}}{\partial y^l_1}\frac{\partial^3 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1 \partial y^k_1} + \frac{1}{8}\hat{g}^{jh}\frac{\partial^4 \hat{g}^{is}}{\partial y^j_1 \partial y^i_1 \partial y^k_1 \partial y^l_1}]
\end{aligned}
\tag{30}
$$

If we differentiate (30) with respect to $y_2^\nu$ and then contract it with $y_2^\nu$ and replace this result to (30), we gain (27). By applying (24) and (25), we have

$$\begin{aligned}
&Kf^2\check{g}_{\mu\nu} - \check{\mathrm{Ric}}_{\mu\nu} - \frac{1}{8}y_2^\beta \frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^h}\frac{\partial^3 F_2^2}{\partial x_2^\beta \partial y_2^\mu \partial y_2^\nu} = \frac{1}{2}\frac{\partial^2 F_2^2}{\partial y_2^\mu \partial y_2^\nu}[-\frac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h}\\
&\quad - \frac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^h \partial x_1^i} + \frac{1}{4}y_1^j \frac{\partial^2 \hat{g}^{ih}}{\partial x_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^h} + \frac{1}{4}y_1^j \frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial^2 f^2}{\partial x_1^j \partial x_1^h} - \frac{1}{2}\hat{G}^j \frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^s}\\
&\quad - \frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji} + \frac{1}{4}\frac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i} - \frac{1}{2}\frac{1}{f^2}y_1^h \frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial \hat{g}^{is}}{\partial y_1^i} + \frac{1}{4}\hat{G}^i_j \frac{\partial \hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}\\
&\quad + \frac{1}{4}\hat{G}^j_i \frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}] + \frac{1}{2}\frac{1}{f^2}y_1^h \frac{\partial f^2}{\partial x_1^h}\check{G}^\alpha_{\mu\nu\alpha} - \frac{1}{4}\frac{\partial \hat{g}^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}[\check{G}^\beta_{\mu\nu}\frac{\partial F_2^2}{\partial y^\beta} + \check{G}^\beta_\mu \frac{\partial F_2^2}{\partial y_2^\beta \partial y_2^\nu}\\
&\quad + \check{G}^\beta \frac{\partial^3 F_2^2}{\partial y_2^\beta \partial y_2^\mu \partial y_2^\nu} + \check{G}^\beta_\nu \frac{\partial F_2^2}{\partial y_2^\beta \partial y_2^\mu}] + \frac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^2 F_2^4}{\partial y_2^\mu \partial y_2^\nu}[-\frac{1}{16}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial \hat{g}^{js}}{\partial y_1^i}\\
&\quad + \frac{1}{8}\hat{g}^{jh}\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}]
\end{aligned}$$

Let us different of the above equation with respect to $y_1^k$ and contract it by $y_1^k$, then replace this result to (31), we obtain

$$\begin{aligned}
&Kf^2\check{g}_{\mu\nu} - \check{\mathrm{Ric}}_{\mu\nu} = \check{g}_{\mu\nu}[-\frac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} - \frac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^h \partial x_1^i} - \frac{1}{2}\hat{G}^j \frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^s}\\
&\quad - \frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji} + \frac{1}{4}\frac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i} + \frac{1}{4}\hat{G}^i_j \frac{\partial \hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} + \frac{1}{4}\hat{G}^j_i \frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}]\\
&\quad + 3\check{g}_{\mu\nu}F_2^2[\frac{1}{8}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial \hat{g}^{js}}{\partial y_1^i} - \frac{1}{4}\hat{g}^{jh}\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}] + \frac{1}{2}\frac{1}{f^2}y_1^h \frac{\partial f^2}{\partial x_1^h}\check{G}^\alpha_{\mu\nu\alpha}
\end{aligned} \tag{31}$$

Differentiating of (31) again with respect to $y_1^l$ and then contracting it with $y_1^l$ and inserting into (31). Therefore by Lemma 3.2, we obtain equation (28). □

The converse of the above theorem is established by placing an additional condition.

**Theorem 4.3.** *Let $\mathbb{F} = \mathbb{F}_1 \times_f \mathbb{F}_2$ be a warped product Finsler space. The warped product Finsler metric $F$ is an Einstein metric of constant $K$ if equations (27), (28) and the following condition hold:*

$$\begin{aligned}
F_2^2[\frac{1}{2}y_1^j \frac{\partial^2 \hat{g}^{ih}}{\partial x_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^h} &+ \frac{1}{2}y_1^j \frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial^2 f^2}{\partial x_1^h \partial x_1^j} - \frac{1}{f^2}y_1^h \frac{\partial f^2}{\partial x_1^h}\frac{\partial \hat{g}^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}] =\\
&\frac{\partial \hat{g}^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}\check{G}^\beta \frac{\partial F_2^2}{\partial y_2^\beta} - \frac{1}{2}y_2^\beta \frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^h}\frac{\partial F_2^2}{\partial x_2^\beta}\\
&-\frac{\partial f^2}{\partial x_1^s}\frac{\partial f^2}{\partial x_1^h}F_2^4[\frac{1}{4}\hat{g}^{jh}\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i} - \frac{1}{8}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial \hat{g}^{js}}{\partial y_1^i}]\,.
\end{aligned} \tag{32}$$

*Proof.* Let us contract Eq. (32) with $\frac{1}{4}$ and then different it with respect to $y_1^k$ and $y_2^\nu$ . Also, differentating (28) with respect to $y_1^k$ and contracting it with $-\frac{1}{4}\frac{\partial F_2^2}{\partial y_2^\nu}$. By inserting these results into (23), we obtain $\mathrm{Ric}_{k\nu} = 0$.

Now, we different (28) with respect to $y_1^k$ and $y_1^l$ and then contract it with $-\frac{1}{4}F_2^2$ and different also Eq. (32) with respect to $y_1^k$ and $y_1^l$. Then replace these results into (22). By using (27) we get $\mathrm{Ric}_{kl} = K\hat{g}_{kl}$.

Finally, differentiating (32) with respect to $y_2^\mu$ and $y_2^\nu$ and using (28) and (24) we obtain $\mathrm{Ric}_{\mu\nu} = Kf^2\check{g}_{\mu\nu}$. Therefore the warped product Finsler metric $F$ is an Einstein metric of constant $K$. □

Applying Theorems (4.2), (4.3) and Lemma (3.1), we may derive new results about special warped product Finsler spaces. As an example, we have the following Corollary.

**Corollary 4.4.** *The warped product Finsler space $\mathbb{F} = \mathbb{F}_1 \times_f \mathbb{F}_2$ when the base space is a Riemannian and the fiber space is a locally Minkowski space, is Einstein of constant $K$ if and only if the following equations are satisfied*

$$K\hat{g}_{kl} = \hat{\mathrm{Ric}}_{kl} + \frac{1}{2}\frac{\partial \ln f^2}{\partial x_1^j}\hat{G}^j_{kl} - \frac{1}{2}\frac{\partial^2 \ln f^2}{\partial x_1^k \partial x_1^l},$$

$$Kf^2 = \frac{1}{4}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) - \frac{1}{2}\hat{\Delta}_h f^2.$$

**Corollary 4.5.** *Let $(M, F)$ be a warped product Finsler space where the base space is a locally Minkowski space. Consider $F$ is an Einstein metric with constant $K$. Then the constant $K$ depends on the warping function and the Ricci tensor of the fiber space is given by*

$$\check{\mathrm{Ric}}_{\mu\nu} = \frac{1}{2}\check{g}^{\mu\nu}\left[\hat{\Delta}_h f^2 - \frac{1}{2}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) - \hat{g}^{kl} f^2 \frac{\partial \ln f^2}{\partial x_1^k \partial x_1^l}\right] - y_1^h \frac{\partial \ln f^2}{\partial x_1^h}\check{G}^\alpha_{\mu\nu\alpha}.$$

In Finsler geometry, the warped product Finsler space is Riemannian if and only if the fiber and the base space are Riemannian [2]. Thus we can state the following Corollary.

**Corollary 4.6.** *Let $(M, F)$ be a warped product Finsler space that is Riemannian space. The warped product Finsler metric $F = F_1 \times_f F_2$ is an Einstein metric of constant $K$ if and only if the fiber space is an Einstein of constant $K_2$ i.e $\mathrm{Ric}_{\mu\nu} = K_2\check{g}_{\mu\nu}$ and the followings hold:*

$$\hat{\mathrm{Ric}}_{kl} = K\hat{g}_{kl} - \frac{1}{2}\frac{\partial \ln f^2}{\partial x_1^j}\hat{G}^j_{kl} + \frac{1}{2}\frac{\partial^2 \ln f^2}{\partial x_1^k \partial x_1^l},$$

$$K_2 - Kf^2 = \frac{1}{2}\hat{\Delta}_h f^2 - \frac{1}{4}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2).$$

This corollary exactly holds when the base space is Riemannian and the fiber space is Berwald space.

In Riemannian geometry, the static vacuum Einstein equations on a manifold $M$ are given by

$$f\mathrm{Ric} = D^2 f, \quad \Delta f = 0, \tag{33}$$

where $f\colon M \to \mathbb{R}$ is a positive function, $D^2$ is the Hessian and $\Delta = \mathrm{tr}D^2$ is the Laplacian on $(M, g)$. These equations are equivalent to the statement that the manifold $N = M \times_f S^1$ or $N = M \times_f \mathbb{R}$ with Riemannian metric of the form $g_N = g_M + f^2 \mathrm{d}t^2$ is Ricci-flat, i.e. $\mathrm{Ric}_{g_N} = 0$. These equations have been extensively studied in the physics literature on classical relativity, where the solutions represent space-times outside regions of matter which are translation and reflection invariant in the time direction $t$. However, many of the global properties of solutions have not been rigorously examined, either from mathematical or physical points of view, for example see [12].

**Lemma 4.7.** *The static vacuum Einstein equations on a Finsler manifold $M$ are given by*

$$\hat{\mathrm{Ric}}_{kl} = \frac{1}{2}\frac{\partial^2 \ln f^2}{\partial x_1^k \partial x_1^l} - \frac{1}{2}\frac{\partial \ln f^2}{\partial x_1^j}\hat{G}^j_{kl},$$

$$\hat{\Delta}_h f^2 = \frac{1}{2}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2).$$

*Solutions of these equations define a Ricci-flat warped product Finsler manifold $N$, of the form $N = M \times_f S^1$ or $N = M \times_f \mathbb{R}$.*

*Proof.* Using (22), (23) and (24). □

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