

ON LOCALLY DUALY FLAT SPECIAL FINSLER (α, β) -METRICS

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ABSTRACT. In this paper, we characterize locally dually flat (α, β) -metrics $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}, (m \neq 0, -1)$, with isotropic S -curvature and scalar flag curvature and show that these metrics reduce to locally Minkowskian metrics.

1. INTRODUCTION

The notion of dually flat metrics was first introduced by S.-I. Amari and H. Nagaoka [2] when they studied the information geometry on Riemannian spaces. Later on, Z. Shen extends the notion of dually flatness to Finsler metrics [10]. A geodesic curve $c = c(t)$ of a Finsler metric $F = F(x, y)$ on a smooth manifold M is given by $\ddot{c}^i(t) + 2G^i(c(t), \dot{c}(t)) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients given by

$$(1) \quad G^i = \frac{1}{4} g^{il} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \}.$$

A Finsler metric $F = F(x, y)$ on a manifold is locally dually flat if at every point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x, y)$ is a C^∞ scalar function on TM_0 satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system. In [10], it is proved that a Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is locally dually flat if and only if it satisfies the following PDE

$$(2) \quad [F^2]_{x^k y^l} y^k - 2 [F^2]_{x^l} = 0.$$

In this case, H is given by $H = -\frac{1}{6} [F^2]_{x^m} y^m$.

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It is known that a Riemannian metric $F = \sqrt{a_{ij}(x)y^i y^j}$ is locally dually flat if and only if in an adapted coordinate system, $a_{ij}(x) = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x)$, where $\psi = \psi(x)$ is a C^∞ function [2, 1]. The first example of non-Riemannian dually flat metrics is Funk metric, given in [10] as follows

$$(3) \quad F(x, y) = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)})}{(1 - |x|^2)} \pm \frac{\langle x, y \rangle}{(1 - |x|^2)}.$$

Above metric is defined on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$. Also it is locally projectively flat with constant flag curvature $K = -\frac{1}{4}$. More general, we have the following

Example 1. [5] Let $U \subset \mathbb{R}^n$ be a strongly convex domain, namely, there is a Minkowski norm $\phi(y)$ on $\mathbb{R}^n$ such that

$$U := \{y \in \mathbb{R}^n \mid \phi(y) < 1\}.$$

Define $\Theta = \Theta(x, y) > 0, y \neq 0$ by

$$x + \frac{y}{\Theta} \in \partial U,$$

$y \in T_x U = \mathbb{R}^n$. It is easy to show that Θ is a Finsler metric satisfying

$$(4) \quad \Theta_{x^k} = \Theta \Theta_{y^k}.$$

Using equation (4), it is easy to verify that $\Theta = \Theta(x, y)$ satisfies equation (2). Thus it is locally dually flat on U . Θ is called Funk metric on U . It is easy to see that Funk metric is of constant flag curvature $K = -\frac{1}{4}$. Also, Θ is of constant S -curvature, $S = \frac{n+1}{2}\Theta$. In particular, when $U = \mathbb{B}^n(1)$, the Funk metric is just the metric in the form of equation (3).

In fact, every locally dually flat and projectively flat metric on an open subset in $\mathbb{R}^n$ must be either a Minkowski metric or a Funk metric satisfying (4) after normalization.

A Finsler metric F is called projectively flat if F is projectively equivalent to a Minkowski/Euclidean metric. In this case, all geodesics of F are straight lines, namely, we can characterize geodesics of F as $\sigma(t) := f(t)a + b$ for some constant vectors $a, b \in \mathbb{R}^n$.

In [6], it is shown that a Finsler metric F on a manifold M is projectively flat if and only if F satisfies the following

$$F_{x^k y^l} y^k - F_{x^l} = 0.$$

In this case,

$$G^i = P(x, y) y^i,$$

with $P = \frac{F_{x^k} y^k}{2F}$. We call P the projective factor of F .

Lemma 1. [4] Let $F = F(x, y)$ be a Finsler metric on an open set $U \subset \mathbb{R}^n$. Then F is locally dually flat and projectively flat on U if and only if $F_{x^k} = C F F_{y^k}$, where C is a constant.

For a Finsler metric F , the Riemann curvature $R_y = R_k^i \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R_k^i := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial x^k} + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

The Ricci curvature is the trace of the Riemann curvature, $\text{Ric} := R_m^m$. The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald. Riemannian metrics of constant sectional curvature were classified by E. Cartan a long time ago. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However, the local metric structure of a Finsler metric with constant flag curvature is much more complicated. For a flag $\{P, y\}$ in $T_x M$, where $P \subset T_x M$ is a tangent plane containing y , the flag curvature $K(x, y, P)$ is defined by

$$K(x, y, P) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)g_y(y, u)},$$

where $u \in P$ such that $P = \text{span}\{y, u\}$. A Finsler metric F is said to be of scalar flag curvature if $K(x, y, P) = K(x, y)$ is independent of P containing $y \in T_x M$. F is said to be of isotropic scalar flag curvature if $K(x, y, P) = K(x)$ and of constant flag curvature if $K(x, y, P) = \text{constant}$.

The S -curvature $S = S(x, y)$ in Finsler geometry is introduced by Shen [11] as a non-Riemannian quantity, defined as

$$S(x, y) = \frac{d}{dt} [\tau(\sigma(t), \dot{\sigma}(t))]_{|t=0},$$

where $\tau = \tau(x, y)$ is a scalar function on $T_x M \setminus \{0\}$, called distortion of F and $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$.

A Finsler metric F is called of isotropic S -curvature if

$$S = (n + 1)cF$$

for some scalar function $c = c(x)$ on M . One of the fundamental problems in Riemann-Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature with isotropic S -curvature.

In [14], the author studied Finsler metrics in the form $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ with non-zero constants ϵ and k and found that there is no locally dually flat Finsler metric in this form with constant flag (even of scalar flag curvature) or isotropic S -curvature unless it is Minkowskian. Several geometer [12, 9, 8] also have studied different class of (α, β) -metrics and found that these class of locally dually flat Finsler (α, β) -metric with isotropic S -curvature and constant flag curvature again reduces to Minkowskian. These facts inspire us to study more general Finsler (α, β) -metrics $F = \frac{(\alpha + \beta)^{m+1}}{\alpha^m}$, with isotropic S -curvature and flag curvature.

This class of (α, β) -metrics $F = \frac{(\alpha + \beta)^{m+1}}{\alpha^m}$ contains Randers metric $F = \alpha + \beta$ for $m = 0$; Riemannian metric $F = \alpha$ for $m = -1$; Matsumoto metric $F =$

$\frac{\alpha^2}{(\alpha-\beta)}$, if we replace β by $-\beta$ and take $m = -2$; and Square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ for $m = 1$.

More precisely we have the following theorems.

Theorem 1. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ ($m \neq 0, -1$) be the (α, β) -metric on a manifold M and F is locally dually flat metric with isotropic S -curvature.*

- (i) *If $\tau[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0$, then F is locally projectively flat in adapted coordinate system with $G^i = C\tau\beta y^i$.*
- (ii) *If $s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0$, then F is locally projectively flat in adapted coordinate system with $G^i = 0$.*

Theorem 2. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ ($m \neq 0, -1$) be a non-Riemannian (α, β) -metric on a manifold M . Then F is locally dually flat with isotropic S -curvature $S = (n+1)cF$ and scalar flag curvature if and only if it is locally Minkowskian.*

2. PRELIMINARIES

Let M be an n -dimensional C^∞ -manifold. $T_x M$ denotes the tangent space of M at x and the tangent bundle TM is the disjoint union of tangent spaces $TM := \bigcup_{x \in M} T_x M$. We denote the elements of TM by (x, y) where $y \in T_x M$. Let $TM_0 = TM \setminus \{0\}$.

A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ with the following properties:

- (i) F is C^∞ on TM_0 ,
- (ii) F is positively 1-homogeneous, and
- (iii) the Hessian of $\frac{F^2}{2}$, with elements $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$, is positive definite on TM_0 .

The pair (M, F) is then called a Finsler space.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric, $\beta = b_i y^i$ be a 1-form and let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\phi = \phi(s)$ is a positive C^∞ function defined in a neighbourhood of the origin $s = 0$. It is well known that $F = \alpha\phi(s)$ is a Finsler metric for any α and β with $b = \|\beta\|_\alpha < b_0$ if and only if $\phi(s) > 0$, $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ ($|s| \leq b < b_0$).

Let G_α^i denote the spray coefficients of α given by

$$(5) \quad G_\alpha^i = \frac{1}{4} a^{il} \{ [\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^l} \},$$

where $(a^{ij}) = (a_{ij})^{-1}$. In view of equations (1) and (5), we have

$$(6) \quad G^i = G_\alpha^i + R y^i + Q^i,$$

where

$$(7) \quad R = \alpha^{-1} \Theta \{-2Q\alpha s_0 + r_{00}\},$$

$$\begin{aligned}
(8) \quad Q^i &= \alpha Q s_0^i + \Psi \{-2Q\alpha s_0 + r_{00}\} b^i, \\
\Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\
\Psi &= \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}$$

Consider the following notations [11]

$$\begin{aligned}
r_{ij} &= \frac{1}{2} \{b_{i;j} + b_{j;i}\}, & r_j^i &= a^{ih} r_{hj}, & r_j &= b_i r_j^i, & s_{ij} &= \frac{1}{2} \{b_{i;j} - b_{j;i}\}, \\
s_j^i &= a^{ih} s_{hj}, & s_j &= b_i s_j^i, & b^i &= a^{ih} b_h, & b^2 &= b^i b_i,
\end{aligned}$$

where $b_{i;j}$ is the covariant derivative of b_i with respect to Levi-Civita connection of α .

In [13], Q. Xia have studied a class of locally dually flat (α, β) -metrics and obtained the following results.

Theorem 3. [13] *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β) -metric on n -dimensional manifold M where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i y^i \neq 0$. Suppose that F is not Riemannian and $\phi'(0) \neq 0$. Then F is locally dually flat on M if and only if α, β and $\phi = \phi(s)$ satisfy*

$$\begin{aligned}
(9) \quad s_{l0} &= \frac{1}{3}(\beta\theta_l - \theta b_l), \\
r_{00} &= \frac{2}{3}\theta\beta + [\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\tau\beta^2, \\
G_\alpha^l &= \frac{1}{3}[2\theta + (3k_1 - 2)\tau\beta]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l, \\
\tau[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] &= 0,
\end{aligned}$$

where $\tau := \tau(x)$ is a scalar function, $\theta := \theta_i(x)y^i$ is a 1-form on M , $\theta^l := a^{lm}\theta_m$ and $k_1 := \Pi(0)$, $k_2 := \frac{\Pi'(0)}{Q(0)}$, $k_3 = \frac{1}{6Q(0)^2}[3Q''(0)\Pi'(0) - 6\Pi'(0)^2 - Q(0)|\Pi'''(0)|]$, $Q := \frac{\phi'}{\phi - s\phi'}$, $\Pi := \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}$.

Corollary 1. *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β) -metric on n -dimensional manifold M with the same assumptions as in above Theorem 3. If ϕ satisfies*

$$s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0,$$

then F is locally dually flat on M if and only if α and β satisfy

$$\begin{aligned}
s_{l0} &= \frac{1}{3}(\beta\theta_l - \theta b_l), \\
r_{00} &= \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2], \\
G_\alpha^l &= \frac{1}{3}[2\theta y^l + \theta^l \alpha^2].
\end{aligned}$$

In [3], Cheng-Shen have studied the class of (α, β) -metrics of non-Randers type $\phi \neq t_1\sqrt{1+t_2s^2}+t_3s$ with isotropic S -curvature and obtained the following.

Theorem 4. [3] *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian (α, β) -metric on a manifold and $b = \|\beta_x\|_\alpha$. Suppose that $\phi \neq t_1\sqrt{1+t_2s^2}+t_3s$ for any constant $t_1 > 0, t_2$ and t_3 . Then F is of isotropic S -curvature $S = (n+1)cF$, if and only if one of the following holds.*

(i) β satisfies

$$r_{ij} = \epsilon\{b^2a_{ij} - b_ib_j\}, s_j = 0,$$

where $\epsilon = \epsilon(x)$ is a scalar function and $c = c(x)$ satisfies

$$(10) \quad \Phi = -2(n+1)k \frac{\phi\Delta^2}{b^2 - s^2},$$

where k is a constant. In this case $S = (n+1)cF$ with $c = k\epsilon$.

(ii) β satisfies

$$r_{ij} = 0, s_j = 0.$$

In this case, $S = 0$, regardless of choices of a particular ϕ .

3. LOCALLY DUALY FLATNESS AND ISOTROPIC S -CURVATURE

For the Finsler metric $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ ($m \neq 0, -1$), we obtain

$$\begin{aligned} \phi &= (1+s)^{m+1}, \\ \phi' &= (m+1)(1+s)^m, \\ \phi'' &= m(m+1)(1+s)^{m-1}, \\ \phi''' &= m(m^2-1)(1+s)^{m-2}, \end{aligned}$$

$$\begin{aligned}
\Pi &= -\frac{2m^2 + 3m + 1}{(1+s)(-1+sm)}, \\
\Pi' &= \frac{2m^3 + m^2 + 4m^3s - 2m + 6sm^2 - 1 + 2sm}{(1+s)^2(-1+sm)^2}, \\
\Pi'' &= \frac{-2(1+2m+6s^2m^4+9s^2m^3+3s^2m^2+6sm^4+2m^4-3sm)}{(1+s)^3(-1+sm)^3} \\
&\quad + \frac{-2(m^3-6sm^2+3m^3s)}{(1+s)^3(-1+sm)^3}, \\
\Pi''' &= \frac{6(-1+8sm^5-2m+2m^5+6s^2m^4-12s^2m^3-6s^2m^2+4sm^4)}{(1+s)^4(-1+sm)^4} \\
&\quad + \frac{6(12s^2m^5+8m^5s^3+12m^4s^3+m^4+4s^3m^3+4sm+8sm^2)}{(1+s)^4(-1+sm)^4}, \\
Q &= -\frac{m+1}{-1+sm}, \\
Q' &= \frac{(m+1)m}{(-1+sm)^2}, \\
Q'' &= -2\frac{(m+1)m^2}{(-1+sm)^3}, \\
k_1 &= 2m^2 + 3m + 1, \\
k_2 &= 2m^2 - m - 1, \\
k_3 &= -m(4m^3 - 4m^2 - m + 1).
\end{aligned}$$

By using above values in Theorem (3), we have the following two lemmas.

Lemma 2. *If $s(k_2 - k_3s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') = 0$, then F is locally dually flat on M if and only if α, β and $\phi = \phi(s)$ satisfy the following equations*

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$$

$$(11) \quad r_{00} = \frac{2}{3}\theta\beta + [\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3b^2)\tau\beta^2,$$

$$(12) \quad G_\alpha^l = \frac{1}{3}[2\theta + (3k_1 - 2)\tau\beta]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2b^l,$$

$$\tau[s(k_2 - k_3s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0,$$

where $k_1 = 2m^2 + 3m + 1, k_2 = 2m^2 - m - 1, k_3 = -m(4m^3 - 4m^2 - m + 1)$.

Lemma 3. *If $s(k_2 - k_3s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0$, then F is locally dually flat on M if and only if α and β satisfy*

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$$

$$r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2],$$

$$G_\alpha^l = \frac{1}{3}[2\theta y^l + \theta^l \alpha^2].$$

Further the given Finsler metrics $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$, we have the following proposition.

Proposition 1. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ ($m \neq 0, -1$) be the (α, β) -metric on a manifold M . Then F is of isotropic S -curvature $S = (n+1)cF$ if and only if β satisfies*

$$r_{ij} = 0, \quad s_j = 0.$$

Proof. Let $\phi = \phi(s)$ be a positive C^∞ function defined on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$(13) \quad \Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'',$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q'$$

and

$$Q := \frac{\phi'}{\phi - s\phi'}.$$

Now equation (13) can be written as

$$(14) \quad \Phi = -(Q - sQ')(n+1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\}.$$

We suppose that the case (i) of Theorem (4) holds. For the metric $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$, we have

$$\Delta = \frac{1 + s - ms - 2ms^2 - s^2m^2 + m^2b^2 + mb^2}{(ms - 1)^2}.$$

It follows that $(ms - 1)^2 \Delta$ is a polynomial in s of degree 2. On the other hand, we have

$$(15) \quad \phi\Delta^2 = \frac{\phi(1 + s - ms - 2ms^2 - s^2m^2 + m^2b^2 + mb^2)^2}{(ms - 1)^4}.$$

Substituting equation (15) into equation (10), we get

$$(16) \quad (b^2 - s^2)(ms - 1)^4\Phi = -2(n+1)k(1+s)^{m+1}(1 + s - ms - 2ms^2 - s^2m^2 + m^2b^2 + mb^2)^2.$$

Now by considering another form of Φ defined by equation (14), we have

$$\begin{aligned} \Phi = & \frac{2m^4nb^2s - 2m^4ns^3 + 4m^3nb^2s - 6m^3ns^3 - m^3nb^2 - 2m^3b^2s + 2m^2nb^2s}{(ms - 1)^4} \\ & - \frac{m^3ns^2 - 4m^2ns^3 - 2b^2m^3 - 2m^2nb^2 - 2m^2b^2s + 3m^2ns^2 - 2b^2m^2 + nb^2m}{(ms - 1)^4} \\ & - \frac{3s^2m^2 + 4mns^2 + 3m^2ns + 3m^2s + 3s^2m}{(ms - 1)^4} \\ & - \frac{-mn + 2mns - ns - m + 2ms - n - s - 1}{(ms - 1)^4}. \end{aligned}$$

Equation (16) can be rewritten as

$$\begin{aligned} (b^2 - s^2)(ms - 1)^4\Phi = & -2b^4m^2n - b^4nm - 2b^4m^2 - 2b^4m^3 - nb^2 - nb^2m \\ & - b^4m^3n - b^2 - b^2m + s(2b^4m^4n + 4b^4m^3n - 2b^4m^3 + 2b^4m^2n - 2b^4m^2 \\ & + 3m^2nb^2 + 3b^2m^2 + 2nb^2m + 2b^2m - nb^2 - b^2) + s^2(2b^2m^3 + 5m^2nb^2 \\ & + 5b^2m^2 + 5nb^2m + 3b^2m + mn + m + n + 1) + s^3(-4m^4nb^2 - 10m^3nb^2 \\ & + 2b^2m^3 - 6m^2nb^2 + 2b^2m^2 - 3m^2n - 3m^2 - 2mn - 2m + n + 1) \end{aligned} \quad (17)$$

In view of equation (16) and (17), $(b^2 - s^2)(ms - 1)^4\Phi$ does have same degree of polynomial in s and b only if $m = 0$ which contradicts our assumptions. Therefore case (ii) of Theorem (4) holds. In this case, we have

$$(18) \quad r_{00} = 0,$$

$$(19) \quad s_j = 0.$$

□

4. PROJECTIVELY FLATNESS AND SCALAR FLAG CURVATURE

In this section, we have studied projective flatness of locally dually flat Finsler metric F for both Lemma 2 and 3.

For lemma 2 of locally dually flat:

Proposition 2. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ ($m \neq 0, -1$) be the (α, β) -metric on a manifold M . Then locally dually flat metric F with isotropic S -curvature is locally projectively flat in adapted coordinate system with $G^i = C\tau\beta y^i$.*

Proof. From Proposition (1), we have $r_{ij} = 0$. Then using equation (11), we get $[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 = [-\frac{2}{3}\theta - \frac{1}{3}(3k_2 - 2 - 3k_3b^2)\tau\beta]\beta$. Since α^2 is irreducible polynomial of y^i , we conclude that

$$(20) \quad \tau + \frac{2}{3}(b^2\tau - \theta_l b^l) = 0$$

and

$$2\theta + (3k_2 - 2 - 3k_3b^2)\tau\beta = 0.$$

Now contracting equation (9) with b^l , we get

$$s_0 = \frac{1}{3}(\beta b^l \theta_l - \theta b^2).$$

Since $s_0 = 0$ the above equation can be written as

$$(21) \quad b^l \theta_l = \frac{\theta b^2}{\beta}.$$

Using equation (20) and (21), we have

$$\theta = \frac{3+2b^2}{2b^2}\tau\beta \quad \text{and} \quad \theta_l = \frac{3+2b^2}{2b^2}\tau b_l.$$

Using the values of θ and θ_l in equation (9), we get $s_{ij} = 0$. Thus β is closed.

Let

$$(22) \quad \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 = \frac{1}{3} \left[\left(\frac{3+2b^2}{2b^2} - 1 \right) \tau\beta \right] y^l$$

and

$$(23) \quad \frac{1}{2}k_3\tau\beta^2b^l = \frac{1}{2}k_3b^2\tau\beta y^l.$$

Using equation (22), (23), and (12), we get

$$G_\alpha^l = C\tau\beta y^l,$$

where $C = \left[\frac{1+k_1}{b^2} + \frac{1}{2b^2} + \frac{k_3b^2}{2} \right]$ is a constant. Thus α is projectively flat.

Now we have $r_{ij} = 0$, $s_j = 0$ and $s_{ij} = 0$, using these values in equations (6), (7), and (8) we obtain $R = 0$ and $Q^i = 0$. Thus we have $G^i = Py^i$, where $P = C\tau\beta$. Thus we can say that locally dually flat metric F with isotropic S -curvature is locally projectively flat in adapted coordinate system. $\square$

For lemma 3 of locally dually flat:

Proposition 3. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}(m \neq 0, -1)$ be a locally dually flat non-Randers type (α, β) -metric on a manifold M . Suppose that F is of isotropic S -curvature $S = (n+1)cF$, where $c = c(x)$ is a scalar function on M . Then F is a locally projectively flat in adapted coordinate systems with $G^i = 0$.*

Proof. We have

$$(24) \quad s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$$

$$(25) \quad r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2],$$

$$(26) \quad G_\alpha^l = \frac{1}{3}[2\theta y^l + \theta^l \alpha^2].$$

By (18) and (25), we obtain

$$(\theta_l b^l) \alpha^2 = \theta \beta.$$

Since α^2 is an irreducible polynomial the above equation reduces to

$$\theta = 0,$$

and

$$\theta_l b^l = 0.$$

Then equations (24), (25), and (26) become

$$(27) \quad \begin{aligned} s_{l0} &= 0, \\ G_\alpha^m &= 0, \\ r_{00} &= 0. \end{aligned}$$

By equations (19) and (27), we get $s_0 = 0$ and $s_0^l = 0$ respectively. Thus by equation (6), we get $G^i = 0$. $\square$

Proposition 4. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ ($m \neq 0, -1$) be locally projectively flat with zero flag curvature. If $\tau = 0$, then F is locally Minkowskian.*

Proof. Let us assume that F is locally projectively flat, so that in local coordinate system the spray coefficients of F are in the form $G^i = P y^i$, where in our case $P = C\tau\beta$. It is known that if the spray coefficients of F are in the form $G^i = P y^i$, then F is of scalar flag curvature with

$$(28) \quad K = \frac{P^2 - P_{x^k} y^k}{F^2} = \frac{C^2 \tau^2 \beta^2 - C \tau_{x^k} \beta y^k - C \beta_{x^k} \tau y^k}{F^2}.$$

$\square$

Lemma 4. *Suppose that $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ ($m \neq 0, -1$) is projectively flat with constant flag curvature $K = \lambda = \text{constant}$, then $K = 0$.*

Proof. From equation (28), we have $\lambda(\alpha + \beta)^{2m+2} = (C^2 \tau^2 \beta^2 - C \tau_{x^k} \beta y^k - C \beta_{x^k} \tau y^k) \alpha^{2m}$. Hence $\lambda [\alpha^{2m+2} + \binom{2m+2}{1} \alpha^{2m+1} \beta + \binom{2m+2}{2} \alpha^{2m} \beta^2 + \dots, \beta^{2m+2}] = (C^2 \tau^2 \beta^2 - C \tau_{x^k} \beta y^k - C \beta_{x^k} \tau y^k) \alpha^{2m}$. Comparing the different coefficients of α , we have

- (i) $\lambda \binom{2m+2}{2} = C^2 \tau^2$,
- (ii) $\lambda \alpha \binom{2m+2}{1} = -C \tau_{x^k} y^k$, which is not possible, and
- (iii) $\lambda = 0$.

Case (i) If $\lambda = 0$ then $\tau = 0$. Thus we have $K = 0$. In this case $G^i = G_\alpha^i = 0$.

Case (ii) If $\lambda \binom{2m+2}{2} = C^2 \tau^2$ then $\lambda = \frac{1}{(m+1)(2m+1)} C^2 \tau^2$ which is a function of x only. That is, flag curvature $K = K(x)$. Thus F is Riemannian metric. But we have already assumed that F is non-Riemannian. Thus $\lambda = \frac{1}{(m+1)(2m+1)} C^2 \tau^2$ will be possible only if $\lambda = 0 = \tau$. Again we have $K = 0$. $\square$

Proof of Theorem (2). By Propositions (2) and (3), we conclude that F is dually flat and projectively flat in any adapted coordinate system. By Lemma (1), we have

$$F_{xk} = CF_{y^k}.$$

The spray coefficients $G^i = Py^i$ are given by $P = \frac{1}{2}CF$. Since $G^i = 0$, then $P = 0$ and thus $C = 0$. It implies that $F_{x^k} = 0$ and then F is a locally Minkowskian metric in the adapted coordinated system. This completes the proof. $\square$

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