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ABSOLUTE $|C, \alpha, \beta; \delta|_k$ SUMMABILITY FACTOR OF INFINITE SERIES

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Abstract. In this study, a generalized theorem on a minimal set of sufficient conditions for absolute summable factor has been established by applying a sequence of wider class (quasi-power increasing sequence) and the absolute Cesàro $|C, \alpha, \beta; \delta|_k$ summability for an infinite series. Further a well-known application of the above theorem has been obtained under suitable conditions.

1. Introduction

Let $\sum\limits_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and n^{th} sequence to sequence transformation (mean) of $\{s_n\}$ is given by u_n s.t.

$$u_n = \sum_{k=0}^{\infty} u_{nk} s_k.$$

The series $\sum\limits_{n=0}^{\infty} a_n$ is said to be absolute summable, if

$$\lim_{n \to \infty} u_n = s,$$

and

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}| < \infty.$$

Let τ_n represent the $n^{th}\,(C,1)$ mean of the sequence (na_n), then the series

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$\sum\limits_{n=0}^{\infty} a_n$ is said to be $|C,1|_k$ summable [11] for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n}|\tau_n|^k < \infty.$$

If u_n^α and τ_n^α represent the n^{th} Cesàro mean [1] of order $\alpha > -1$ of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$$

and

$$\tau_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v,$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^\alpha),\ A_{-n}^\alpha = 0 \text{ and } A_0^\alpha = 1 \text{ for } n > 0.$$

The series $\sum\limits_{n=0}^{\infty} a_n$ is said to be $|C,\alpha|_k$ summable for $k \geq 1$ and $\alpha > -1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty}\frac{1}{n}|\tau_n^\alpha|^k < \infty.$$

The series is $|C,\alpha,\beta;\delta|_k$ summable for $k \geq 1,\ \alpha > -1,\ 0 < \beta \leq 1,\ \alpha+\beta > 0$, and $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} n^{(\delta k+k-1)}|u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta}|^k = \sum_{n=1}^{\infty} n^{\delta k-1}|\tau_n^{\alpha,\beta}|^k < \infty.$$

Remark 1. If $\delta = 0$, then $|C,\alpha,\beta;\delta|_k$ summability reduces to $|C,\alpha,\beta|_k$ summability. If we take $\alpha = 1$ & $\beta = 1$, then $|C,\alpha,\beta|_k$ becomes $|C,1,1|_k$ summable and similarly if $\beta = 0$, then $|C,\alpha,\beta|_k$ summability reduces to $|C,\alpha|_k$ summability. If $\alpha = 1$ & $k = 1$, then $|C,\alpha|_k$ summable factor becomes $|C,1|$ summable factor.

For the sequence $\{\tau_n^{\alpha,\beta}\}$ which is n^{th} Cesàro means of $\{na_n\}$, $w_n^{\alpha,\beta}$ can be expressed as [2]

$$w_n^{\alpha,\beta} = \begin{cases} |\tau_n^{\alpha,\beta}|, & \alpha > -1,\ \beta = 1, \\ \max_{1\leq v\leq n}|\tau_v^{\alpha,\beta}|, & \alpha > -1,\ 0 < \beta < 1. \end{cases}$$

2. Known results

Using $|C,\alpha|_k$ summable factor, Bor [3] determined a minimal set of sufficient conditions for an infinite series to be absolute summable. His result can be stated as follows.

Theorem 1. [3] *Let (X_n) be a quasi-f-power increasing sequence for some $\theta\,(0<\theta<1)$. Assume that there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying the following:*

$$\sum n\xi_n X_n = O(1),$$

$$\Delta A_n \le \xi_n,$$

$$|\Delta\lambda_n| \le |A_n|,$$

$$\sum A_n X_n \text{ is convergent for all } n.$$

If the conditions

$$|\lambda_n|X_n = O(1) \text{ as } n\to\infty,$$

$$\sum_{n=1}^{m}\frac{(w_n^{\alpha})^k}{n} = O(X_m) \text{ as } m\to\infty$$

are satisfied, then the series $\sum a_n\lambda_n$ is $|C,\alpha|_k$ summable for $0<\alpha\le 1$ and $k\ge 1$.

3. Main results

Sonker et al. [7, 8, 9, 10] have determined various results for the generalization of the Cesàro summable factor. The aim of the present study is to formulate the problem of generalization of absolute Cesáro summability factor ($|C,\alpha,\beta;\delta|_k$ for $k\ge 1,\ \alpha>-1,\ 0<\beta\le 1,\ \alpha+\beta>0$ and $\delta\ge 0$) for an infinite series. This work will also motivate the researchers interested in theoretical studies of infinite series.

A positive sequence $X=(X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K=K(X,f)\ge 1$ such that $Kf_nX_n\ge f_mX_m$ for all $n\ge m\ge 1$, where $f=[f_n(\theta,\zeta)]=\{n^{\theta}(\log n)^{\zeta},\zeta\ge 0,\ 0<\theta<1\}$ [12]. If $\zeta=0$, then a quasi-θ-power increasing sequence [6] can be obtained.

The results of Bor [3] have been modernized with the help of generalized Cesàro $|C,\alpha,\beta;\delta|_k$ summability and we establish the following theorem.

Theorem 2. *Let (X_n) be a quasi-f-power increasing sequence for some $\theta\,(0<\theta<1)$. Assume that there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying the following:*

$$\sum n\xi_n X_n = O(1), \tag{1}$$

$$\Delta A_n \le \xi_n, \tag{2}$$

$$|\Delta\lambda_n| \le |A_n|, \tag{3}$$

$$\sum A_n X_n \text{is convergent for all } n. \tag{4}$$

If the conditions

$$|\lambda_n|X_n = O(1) \text{ as } n\to\infty, \tag{5}$$

$$\sum_{n=1}^{m}\frac{(w_n^{\alpha,\beta})^k}{n^{1-\delta k}}=O(X_m)\ \textit{as}\ m\to\infty \tag{6}$$

are satisfied, then the series $\sum a_n\lambda_n$ *is* $|C,\alpha,\beta;\delta|_k$ *summable for* $k\geq 1,\ \alpha>-1,\ 0<\beta\leq 1,\ \alpha+\beta>0$ *and* $\delta\geq 0$.

4. Lemmas

The following lemmas have been used to prove the main theorem.

Lemma 1. [5] *If* $0<\beta\leq 1,\ \alpha>-1$ *and* $1\leq v\leq n$, *then*

$$\left|\sum_{p=0}^{v}A_{n-p}^{\beta-1}A_p^{\alpha}a_p\right|=\max_{1\leq m\leq v}\left|\sum_{p=0}^{m}A_{m-p}^{\beta-1}A_p^{\alpha}a_p\right|.$$

Lemma 2. [4] *Let* (X_n) *be a quasi-f-power increasing sequence for some* θ $(0<\theta<1)$. *If* (A_n) *is a ξ-quasi-monotone sequence with* $\Delta A_n\leq\xi_n$ *and* $\sum n\,\xi_n\,X_n<\infty$, *then*

$$\sum_{n=1}^{\infty}nX_n|A_n|<\infty,$$

$$nA_nX_n=O(1)\ \textit{as}\ n\to\infty.$$

5. Proof of Theorem 2

Let $T_n^{\alpha,\beta}$ be the n^{th} (C,α,β) mean of the sequence $(na_n\lambda_n)$. The series is $|C,\alpha,\beta;\delta|_k$ summable, if

$$\sum_{n=1}^{\infty}n^{\delta k-1}|T_n^{\alpha,\beta}|^k<\infty. \tag{7}$$

Applying Abel's transformation and Lemma 1, we have

$$\begin{aligned}T_n^{\alpha,\beta}&=\frac{1}{A_n^{\alpha,\beta}}\sum_{v=1}^{n}A_{n-v}^{\beta-1}A_v^{\alpha}va_v\lambda_v\\
&=\frac{1}{A_n^{\alpha,\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\beta-1}A_p^{\alpha}pa_p+\frac{\lambda_n}{A_n^{\alpha,\beta}}\sum_{v=1}^{n}A_{n-v}^{\beta-1}A_v^{\alpha}va_v \qquad (8)\end{aligned}$$

and

$$\begin{aligned}|T_n^{\alpha,\beta}|&=\frac{1}{A_n^{\alpha,\beta}}\sum_{v=1}^{n-1}|\Delta\lambda_v|\left|\sum_{p=1}^{v}A_{n-p}^{\beta-1}A_p^{\alpha}pa_p\right|+\frac{|\lambda_n|}{A_n^{\alpha,\beta}}\left|\sum_{v=1}^{n}A_{n-v}^{\beta-1}A_v^{\alpha}va_v\right|\\
&=\frac{1}{A_n^{\alpha,\beta}}\sum_{v=1}^{n-1}A_v^{\alpha,\beta}w_v^{\alpha,\beta}|\Delta\lambda_v|+|\lambda_n|w_n^{\alpha,\beta}\\
&=T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}. \qquad (9)\end{aligned}$$

Using Minkowski's inequality,

$$|T_n^{\alpha,\beta}|^k = |T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}|^k < 2^k\Big(|T_{n,1}^{\alpha,\beta}|^k + |T_{n,2}^{\alpha,\beta}|^k\Big). \tag{10}$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1}|T_{n,r}^{\alpha,\beta}|^k < \infty \text{ for } r = 1, 2. \tag{11}$$

By using Hölder's inequality, Abel's transformation and conditions of Lemma 2, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1}|T_{n,1}^{\alpha,\beta}|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1}\frac{1}{(A_n^{\alpha,\beta})^k}\left(\sum_{v=1}^{n-1} A_v^{\alpha,\beta} w_v^{\alpha,\beta}|\Delta\lambda_v|\right)^k \\
&\leq \sum_{n=2}^{m+1} n^{-1-(\alpha+\beta-\delta)k}\sum_{v=1}^{n-1} v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|A_v|\left(\sum_{v=1}^{n-1}|A_v|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|A_v|\sum_{n=v+1}^{m+1}\frac{1}{n^{(\alpha+\beta-\delta)k+1}} \\
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|A_v|\int_v^{\infty}\frac{dx}{x^{(\alpha+\beta-\delta)k+1}} \\
&= O(1)\sum_{v=1}^{m} v|A_v|(w_v^{\alpha,\beta})^k v^{\delta k-1} \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|A_v|)\sum_{r=1}^{v}(w_r^{\alpha,\beta})^k r^{\delta k-1} \\
&\quad +O(1)m|A_m|\sum_{v=1}^{m}(w_v^{\alpha,\beta})^k v^{\delta k-1}
\end{aligned}$$

$$\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Big|(v+1)\Delta|A_v| - |A_v|\Big|X_v + O(1)m|A_m|X_m \\
&= O(1)\sum_{v=1}^{m-1} v|\Delta A_v|X_v + O(1)\sum_{v=1}^{m-1}|A_v|X_v + O(1)m|A_m|X_m \\
&= O(1) \text{ as } m\to\infty,
\end{aligned}$$

and

$$\begin{aligned}\sum_{n=2}^{m} n^{\delta k-1}|T_{n,2}^{\alpha,\beta}|^k &= O(1)\sum_{n=1}^{m}|\lambda_n|(w_n^{\alpha,\beta})^k n^{\delta k-1}\\ &= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}(w_v^{\alpha,\beta})^k v^{\delta k-1}\\ &\quad +O(1)|\lambda_m|\sum_{n=1}^{m}(w_n^{\alpha,\beta})^k n^{\delta k-1}\\ &= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m\\ &= O(1)\sum_{n=1}^{m-1}|A_n|X_n + O(1)|\lambda_m|X_m\\ &= O(1) \text{ as } m\to\infty. \qquad (12)\end{aligned}$$

Collecting (7) - (12), we have

$$\sum_{n=1}^{\infty} n^{\delta k-1}|T_n^{\alpha,\beta}|^k < \infty.$$

Hence the proof of the theorem is complete.

6. Corollaries

Corollary 1. *Let (X_n) be a quasi-f-power increasing sequence for some θ $(0<\theta<1)$. Assume that there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying (1)-(5) and the following:*

$$\sum_{n=1}^{m}\frac{(w_n^{\alpha,\beta})^k}{n} = O(X_m) \text{ as } m\to\infty. \qquad (13)$$

Then the series $\sum a_n\lambda_n$ is $|C,\alpha,\beta|_k$ summable for $k\geq 1,\ \alpha>-1,\ 0<\beta\leq 1$ and $\alpha+\beta>0$.

Proof. Putting $\delta=0$ in Theorem 3.1, we will get (13). We omit the details as the proof is similar to that of Theorem 3.1 and we use (13) instead of (6). □

Corollary 2. *Let (X_n) be a quasi-f-power increasing sequence for some θ $(0<\theta<1)$. Assume that there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying (1)-(5) and the following:*

$$\sum_{n=1}^{m}\frac{(w_n^{\beta})^k}{n} = O(X_m) \text{ as } m\to\infty. \qquad (14)$$

Then the series $\sum a_n\lambda_n$ is $|C,\beta|_k$ summable for $0<\beta\leq 1$ and $k\geq 1$.

Proof. Putting $\alpha = 0$ and $\delta = 0$ in Theorem 3.1, we get (14). We omit the details as the proof is similar to that of Theorem 3.1 and we use (14) instead of (6). $\square$

References

[1] E. Cesàro, *Sur la multiplication des séries*, Bull. Sci. Math., 14(2) (1890), 114-120.

[2] H. Bor, *On a new application of power increasing sequences*, Proc. Est. Acad. Sci., 57(4) (2008), 205-209.

[3] H. Bor, *Some new results on infinite series and Fourier series*, Positivity, 19(3) (2015), 467-473.

[4] H. Bor, *On the quasi monotone and generalized power increasing sequences and their new applications*, J. Classical Anal., 2(2) (2013), 139-144.

[5] L. S. Bosanquet, *A mean value theorem*, J. London Math. Soc., s1-16(3) (1941), 146-148.

[6] L. Leindler, *A new application of quasi power increasing sequences*, Publ. Math. Debr., 58(4) (2001), 791-796.

[7] S. Sonker, A. Munjal, *Absolute summability factor* $\varphi - |C, 1, \delta|_k$ *of Infinite series*, Int. J. Math. Anal., 10(23) (2016), 1129-1136.

[8] S. Sonker, A. Munjal, *Sufficient conditions for triple matrices to be bounded*, Nonlinear Studies, 23(4) (2016), 533-542.

[9] S. Sonker, A. Munjal, *Absolute* $\varphi{-}|C, \alpha, \beta; \delta|_k$ *Summability of Infinite Series*, J. Inequal. Appl., Paper no. 168 (2017), 1-7.

[10] S. Sonker, Xh. Z. Krasniqi, A. Munjal, *A note on absolute Cesàro* $\varphi{-}|C, 1; \delta; l|_k$ *summability factor*, Int. J. Math. Anal., 15(1) (2017), 108-113.

[11] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc., 3(1) (1957), 113-141.

[12] W. T. Sulaiman, *Extension on absolute summability factors of infinite series*, J. Math. Anal. Appl., 322(2) (2006), 1224-1230.

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