

DERIVATIONS OF CONVOLUTION ALGEBRAS ON FINITE PERMUTATION SEMIGROUPS

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ABSTRACT. If $n \in \mathbb{N}$ let S_n be the lexicographically ordered discrete semigroup of permutations of $\{1, \dots, n\}$. Our matter is to seek about the structure and behaviour of derivations of the convolution algebra $l^1(S_n)$. This problem has its own interest even in the finite case and emerges from studies of several kinds of amenability on Banach algebras supported on infinite discrete groups or semigroups.

1. INTRODUCTION

The problem of derivations on convolution algebras has a long-standing interest and even today it is a matter of research (among a huge literature the reader can see [2], [11]). Given a semigroup S and $u, v \in S$ we shall write

$$[uv^{-1}] = \{x \in S : xv = u\} \text{ and } [v^{-1}u] = \{x \in S : vx = u\}.$$

It is known that if S is discrete and contains an infinite pairwise disjoint sequence of sets $X(u_n) = u_n S \cap [u_n u_n^{-1}]$ then $l^1(S, w)$ is not amenable for any weight function w , that is there is always a Banach $l^1(S, w)$ -bimodule $\mathcal{H}$ and a non-inner derivation $D_w : l^1(S, w) \rightarrow \mathcal{H}^*$ (cf. [5], Theorem 1.). In particular, let $S_{\mathbb{N}}$ be the discrete semigroup of functions of the positive integers into itself and let $\mathbb{P}$ be the set of prime positive integers. Given a fix $p \in \mathbb{P}$ and $n \in \mathbb{N}$ we write $u_p(n) = p^{n(p)}$ if $n = p^{n(p)}m$, with $(m : p) = 1$. We can represent $n = \prod_{q \in \mathbb{P}} q^{v_q(n)}$, where $v_q(n) = \max \{s \in \mathbb{N}_0 : q^s \mid n\}$ for each $q \in \mathbb{P}$. Thus, if $\eta(n) \triangleq p^{v_p(n)}$ then $\eta u_p = u_p$ and since u_p is idempotent then $u_p \eta \in X(u_p)$. Further, since $u_p \eta u_p = u_p$ then $(u_p \eta)(p^s) = p^s$ for all $s \in \mathbb{N}_0$. Hence it is readily seen that $\{X(u_p)\}$ is an infinite disjoint sequence of non empty subsets of $S_{\mathbb{N}}$ and $l^1(S_{\mathbb{N}}, w)$ is never amenable.

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Now, remember that a semigroup S is called *inverse semigroup* if for all $v \in S$ there exist a unique element $v^{-1} \in S$ so that $vv^{-1}v = v$ and $v^{-1}vv^{-1} = v^{-1}$. Hence, if S is an inverse semigroup with an infinite set E_S of idempotents $l^1(S, w)$ is never amenable (cf. [5], Corollary 1.). Further, combining this result with the investigation in [4] the algebra $l^1(S, w)$ on an inverse semigroup S is amenable if and only if E_S is finite and every subgroup of S is amenable. Certainly, for all $n \in \mathbb{N}$, by finiteness, each group contained in S_n is amenable but not necessarily an inverse semigroup. For instance, let us consider $(1, 1, 2) \in S_3$, i.e., the function $(1, 1, 2) : 1 \rightarrow 1, 2 \rightarrow 1, 3 \rightarrow 2$. Then $(1, 1, 2)$ has no unique inverse, for

$$\begin{aligned} (1, 1, 2)(1, 3, 1)(1, 1, 2) &= (1, 1, 2), (1, 1, 2)(1, 3, 3)(1, 1, 2) = (1, 1, 2). \\ (1, 3, 1)(1, 1, 2)(1, 3, 1) &= (1, 3, 1), (1, 3, 3)(1, 1, 2)(1, 3, 3) = (1, 3, 3). \end{aligned}$$

It is worth mentioning that any derivation on the group algebra of a discrete group is inner (cf. [7], Theorem 4.). A closer look on the structure and properties of derivations on $l^1(S_n)$ will allow us to derive the results described in Subsection 1.2.

1.1. Notations. If $n \in \mathbb{N}$ and $g \in S_n$, let $\delta_g = \{\delta_{g,h}\}_{h \in X_n}$, where $\delta_{g,h}$ denotes the usual Kronecker symbol. Clearly $\{\delta_g\}_{g \in S_n}$ is a basis of $l^1(S_n)$ and given $D \in L(l^1(S_n))$ there is a unique subset $\{\lambda_g^h\}_{g,h \in S_n}$ of $\mathbb{C}$ such that $D(\delta_g) = \sum_{h \in S_n} \lambda_g^h \delta_h$ for $g \in S_n$. If $\{\delta_g^h\}_{g \in S_n}$ denotes the dual basis of $\{\delta_g\}_{g \in S_n}$ then $\lambda_g^h = \langle D(\delta_g), \delta_h^h \rangle$ for all $g, h \in S_n$. If $\mathbf{m}, \mathbf{p} \in l^1(S_n)$ then there is a unique set $\{F_h^g\}_{g,h \in S_n}$ of linear forms on $l^1(S_n)$ so that

$$(1) \quad \mathbf{m} * \mathbf{p} = \sum_{g \in S_n} \left[\sum_{h \in S_n} \mathbf{m}^h F_h^g(\mathbf{p}) \right] \delta_g.$$

We shall also write S_n as the disjoint union $S_n = \cup_{k=1}^n S_{n,k}$, where $S_{n,k}$ contains the elements $g \in S_n$ so that $\# \text{Im}(g) = k$. The Jacobson radical of $l^1(S_n)$ is denoted by $J(l^1(S_n))$ and the set of derivations on $l^1(S_n)$ by $\mathcal{Z}(l^1(S_n))$.

1.2. Our results. In Section 2 we shall seek about derivations on $l^1(S_n)$. In Theorem 1, (2) we determine the precise conditions on the coefficients λ_g^h in order that D be a derivation; (3) will be related to the innerness matter while (4) will describe the behaviour of the transpose of derivations on the forms F_h^g . In Proposition 1 it is shown how $l^1(S_n)$ can be antihomomorphically embedded into a subalgebra $\mathfrak{F}$ of $M_{n^n}(\mathbb{C})$. In Theorem 2 we shall see that $\mathcal{Z}(l^1(S_n))$ is linearly isomorphic to a quotient of the Lie algebra of matrices $\lambda \in M_{n^n}(\mathbb{C})$ so that $ad_\lambda(\mathfrak{F}) \subseteq \mathfrak{F}$. Here upon in Proposition 2 we will give a complete description of $\mathcal{Z}(l^1(S_2))$. Among other properties, it will be seen that any derivation maps onto the explicitly evaluated Jacobson radical. In [8] it is shown that any element in the image of a bounded derivation on a Banach algebra $\mathcal{U}$ so that $[[\mathcal{U}, \mathcal{U}], \mathcal{U}] = (0)$ is quasi-nilpotent. In [10] it is

proved that any *centralizing* derivation D on a Banach algebra $\mathcal{U}$ maps into the radical. Although those conditions are sufficient by Proposition 2, they are not necessary, even in the finite dimensional context. From Proposition 2 we deduce that

$$[[\delta_{(2,1)}, \delta_{(2,2)}], \delta_{(1,1)}] = \mathbf{q}_0 \neq 0.$$

Further, we know that any non zero derivation D on $l^1(S_2)$ maps onto the radical and is given as $D(\mathbf{m}) = \langle \mathbf{m}, \lambda \rangle \mathbf{q}_0$ for some suitable linear form λ . But

$$[D(\mathbf{m}), \mathbf{m}] = \langle \mathbf{m}, \lambda \rangle (\mathbf{m}^{(1,1)} + 2\mathbf{m}^{(2,1)} + \mathbf{m}^{(2,2)}) \mathbf{q}_0,$$

i.e., $[D(\mathbf{m}), \mathbf{m}]$ is not centralizing since $\mathcal{Z}(l^1(S_2)) = \mathbb{C}\delta_{(1,2)}$.

2. DERIVATIONS ON $l^1(S_n)$

The proof of the following theorem is straightforward:

Theorem 1.

(i) D is a derivation if and only if the following identity holds

$$(2) \quad \lambda_{gh}^l = \sum_{k \in [lh^{-1}]} \lambda_g^k + \sum_{k \in [g^{-1}l]} \lambda_h^k \text{ if } l, g, h \in S_n.$$

(ii) Let D be an inner derivation, say $D = ad_{\mathbf{m}}$ for some $\mathbf{m} \in l^1(S_n)$. If $\mathbf{m} = \sum_{k \in X_n} \mathbf{m}^k \delta_k$ then

$$(3) \quad \lambda_g^h = \sum_{k \in [hg^{-1}]} \mathbf{m}^k - \sum_{k \in [g^{-1}h]} \mathbf{m}^k.$$

(iii) If D is a derivation then

$$(4) \quad D^*(F_h^g) = \sum_{l \in S_n} (\lambda_l^g F_h^l - \lambda_h^l F_l^g) \text{ if } g, h \in S_n.$$

Corollary 1. (cf. [1], Lemma 1.1. (ii)) If $D \in \mathcal{Z}(l^1(S_n))$ then $\varkappa_0(D(\mathbf{m})) = 0$, where $\mathbf{m} \in l^1(S_n)$, $\varkappa_0$ is the augmentation functional, i.e.,

$$\varkappa_0(D(\mathbf{m})) = \langle D(\mathbf{m}), 1 \rangle.$$

Proof. Given $l, h \in S_n$ by (2) we see that

$$(5) \quad \lambda_g^l = \sum_{k \in [lh^{-1}]} \lambda_g^k + \sum_{k \in [g^{-1}l]} \lambda_h^k.$$

As $S_n \circ S_{n,1} \cup S_{n,1} \circ S_n \subseteq S_{n,1}$ if $l \in S_n - S_{n,1}$ then $[g^{-1}l] = \emptyset$. By (5), if we choose $h \in S_{n,1}$ then $[lh^{-1}] = \emptyset$ and $\lambda_g^l = 0$. Now, if $g, h \in S_{n,1}$ we can write

$$\begin{aligned} \sum_{l \in S_{n,1}} \lambda_g^l \delta_l &= D(\delta_g) \\ &= D(\delta_{gh}) \\ &= D(\delta_g * \delta_h) \\ &= D(\delta_g) * \delta_h + \delta_g * D(\delta_h) \\ &= \sum_{l \in S_{n,1}} \lambda_g^l \delta_l + \left[\sum_{l \in S_{n,1}} \lambda_h^l \right] \delta_g, \end{aligned}$$

i.e., $\sum_{l \in S_{n,1}} \lambda_h^l = 0$. Now, given $h \in S_n$ we choose any $g \in S_{n,1}$. Now we get

$$\begin{aligned} D(\delta_g) &= D(\delta_g * \delta_h) \\ &= \left[\sum_{l \in S_{n,1}} \lambda_g^l \delta_l \right] * \delta_h + \delta_g * \sum_{l \in S_n} \lambda_h^l \delta_l \\ &= D(\delta_g) + \left[\sum_{l \in S_n} \lambda_h^l \right] \delta_g \end{aligned}$$

and so $\sum_{l \in S_n} \lambda_h^l = 0$. The general case now follows by simply spanning. $\square$

Proposition 1.

- (i) If $g, h \in S_n$ then $F_h^g = \delta^g \delta_h$ and $F_h^g = \sum_{i \in [h^{-1}g]} \delta^i$.
- (ii) There is a semigroup isomorphism $\hat{n} : S_n \rightarrow S_n$ so that $\hat{n} = \hat{n}^{-1}$ and

$$(6) \quad F_h^g(\delta_f) = F_{\hat{n}(h)}^{\hat{n}(g)}(\delta_{\hat{n}(f)})$$

for all $f, g, h \in S_n$.

- (iii) There is an anti-monomorphism $F : l^1(S_n) \hookrightarrow M_n(\mathbb{C})$.

Proof. (i): Let $\mathbf{m} \in l^1(S_n)$, $l \in S_n$. By using (1) we have

$$\mathbf{m} * \delta_l = \sum_{k \in S_n l} \left[\sum_{f \in [kl^{-1}]} \mathbf{m}^f \right] \delta_k = \sum_{k \in S_n} \left[\sum_{f \in S_n} \mathbf{m}^f F_f^k(\delta_l) \right] \delta_k.$$

If $k \notin S_n l$ then $\sum_{f \in S_n} \mathbf{m}^f F_f^k(\delta_l) = 0$ and we deduce that $F_f^k(\delta_l) = 0$ for $f \in S_n$. If $k \in S_n l$ then

$$\sum_{f \in [kl^{-1}]} \mathbf{m}^f = \sum_{f \in S_n} \mathbf{m}^f F_f^k(\delta_l),$$

and since $\mathbf{m}$ is arbitrary we see that $F_f^k(\delta_l) = 1$ if $fl = k$. Therefore,

$$\begin{aligned}
 F_h^g(\mathbf{m}) &= \sum_{f \in S_n} \mathbf{m}^f F_h^g(\delta_f) \\
 &= \sum_{hf=g} \mathbf{m}^f \\
 &= \sum_{f \in [h^{-1}g]} \mathbf{m}^f \\
 &= \sum_{f \in S_n} \mathbf{m}^f \langle \delta_{hf}, \delta^g \rangle \\
 &= \langle \delta_h * \mathbf{m}, \delta^g \rangle \\
 &= \langle \mathbf{m}, \delta^g \delta_h \rangle.
 \end{aligned}$$

(ii): Let $\widehat{n}(f)(i) = f(n - i + 1)$, $f \in S_n$ and $i \in \{1, \dots, n\}$. It is readily seen that $\widehat{n}$ is a semigroup isomorphism of S_n and $\widehat{n}^{-1} = \widehat{n}$. Moreover, if $\mathbf{m} \in l^1(S_n)$, we can write

$$\begin{aligned}
 (7) \quad F_{\widehat{n}(h)}^{\widehat{n}(g)}(\mathbf{m}) &= \sum_{f \in S_n} \mathbf{m}^{\widehat{n}(f)} \langle \delta_{\widehat{n}(h)\widehat{n}(f)}, \delta^{\widehat{n}(g)} \rangle \\
 &= \sum_{f \in S_n} \mathbf{m}^{\widehat{n}(f)} \langle \delta_{\widehat{n}(hf)}, \delta^{\widehat{n}(g)} \rangle \\
 &= \sum_{f \in S_n} \mathbf{m}^{\widehat{n}(f)} \langle \delta_{hf}, \delta^g \rangle \\
 &= F_h^g(\check{n}(\mathbf{m})),
 \end{aligned}$$

with $\check{n}(\mathbf{m}) = \sum_{f \in S_n} \mathbf{m}^{\widehat{n}(f)} \delta_f$. If $f \in S_n$ we can see that $\check{n}(\delta_f) = \delta_{\widehat{n}(f)}$ and with this fact combined with (7) we get (6).

(iii): Let $F : l^1(S_n) \rightarrow M_{n^n}(\mathbb{C})$ so that $F(\mathbf{p}) = [F_h^g(\mathbf{p})]_{g,h \in S_n}$ if $\mathbf{p} \in l^1(S_n)$, where the upper and lower indexes denote rows and columns, respectively. Then F is clearly linear. If $\mathbf{m}, \mathbf{p} \in l^1(S_n)$ and $g, h \in S_n$, by (1), we have $\delta_h * \mathbf{m} = \sum_{k \in S_n} F_h^k(\mathbf{m}) \delta_k$. Further,

$$\delta_h * \mathbf{m} * \mathbf{p} = \sum_{k \in S_n} F_h^k(\mathbf{m}) \delta_k * \mathbf{p} = \sum_{k \in S_n} F_h^k(\mathbf{m}) \sum_{l \in S_n} F_k^l(\mathbf{p}) \delta_l.$$

Consequently,

$$\begin{aligned}
 F_h^g(\mathbf{m} * \mathbf{p}) &= \langle \mathbf{m} * \mathbf{p}, \delta^g \delta_h \rangle \\
 &= \langle \delta_h * \mathbf{m} * \mathbf{p}, \delta^g \rangle \\
 &= \sum_{k \in S_n} F_h^k(\mathbf{m}) F_k^g(\mathbf{p}) \\
 &= [F(\mathbf{p}) F(\mathbf{m})]_h^g.
 \end{aligned}$$

Since $F(\mathbf{p}) = [\langle \delta_h * \mathbf{p}, \delta_g \rangle]_{g,h \in S_n}$, the injectivity of F is immediate. $\square$

Remark 1. The anti-monomorphism F of Proposition 1 provides a non trivial re-presentation of $l^1(S_n)$ into $\mathbb{C}^{n^n}$. Therefore, it suffices to consider

$$\varrho : l^1(S_n) \rightarrow L(\mathbb{C}^{n^n}), \varrho(\mathbf{m})(x) = x \cdot F(\mathbf{m}).$$

For matrix representations of finite dimensional convolution algebras the reader can see [6].

Theorem 2. *Let $\mathfrak{F} = \text{Im}(F)$ and $\mathcal{D}_n = \{\lambda \in M_{n^n}(\mathbb{C}) : \text{ad}_\lambda(\mathfrak{F}) \subseteq \mathfrak{F}\}$. There is a bijection between $\mathcal{Z}(l^1(S_n))$ and the quotient linear space $\mathcal{D}_n / (\mathcal{D}_n \cap \mathfrak{F}^c)$.*

Proof. By Theorem 1 (iii) and Proposition 1, if $D \in \mathcal{Z}(l^1(S_n))$ then there is $\lambda \in M_{n^n}(\mathbb{C})$ so that $F(D(\mathbf{p})) = \text{ad}_\lambda(F(\mathbf{p}))$. Hence $\lambda \in \mathcal{D}_n$. As $\text{ad}_\lambda(\mathfrak{F}) = (0)$ if and only if $\lambda \in \mathfrak{F}^c$, we get an injection $\Psi : \mathcal{Z}(l^1(S_n)) \hookrightarrow \mathcal{D}_n / (\mathcal{D}_n \cap \mathfrak{F}^c)$. Now, if $\lambda \in \mathcal{D}_n$ let us write $D_\lambda = F^{-1} \circ \text{ad}_\lambda \circ F$ in $L(l^1(S_n))$. Then D_λ is a derivation and the linear mapping $\lambda \rightarrow D_\lambda$ is zero on $\mathcal{D}_n \cap \mathfrak{F}^c$. Finally it is readily seen that the induced mapping $\mathcal{D}_n / (\mathcal{D}_n \cap \mathfrak{F}^c) \rightarrow \mathcal{Z}(l^1(S_n))$ equals Ψ^{-1} . $\square$

Proposition 2.

- (i) *The space $\mathcal{Z}(l^1(S_2))$ is two dimensional.*
- (ii) *If $D \in \mathcal{Z}(l^1(S_2)) - (0)$ then $\text{Im}(D)$ is a non-zero ideal and $\text{Im}(D)^{[2]} = (0)$.*
- (iii) *If $D \in \mathcal{Z}(l^1(S_2))$ then $\text{Im}(D) \subseteq J(l^1(S_2))$.*
- (iv) *Every non-zero derivation within $l^1(S_2)$ maps onto the radical.*

Proof. (i): By applying Theorem 1 (i) or (iii) it is seen that if $\mathbf{m} \in l^1(S_2)$ and D is a derivation on $l^1(S_2)$ then

$$(8) \quad D(\mathbf{m}) = [\alpha \mathbf{m}^{(1,1)} + (\alpha + \beta) \mathbf{m}^{(2,1)} + \beta \mathbf{m}^{(2,2)}] \mathbf{q}_0, \quad \alpha, \beta \in \mathbb{C},$$

where $\mathbf{q}_0 = \delta_{(1,1)} - \delta_{(2,2)}$. In this case we have

$$\begin{aligned} F_{(1,1)}^{(1,1)} &= F_{(2,2)}^{(2,2)} : \mathbf{m} \rightarrow \mathbf{m}^{(1,1)} + \mathbf{m}^{(1,2)} + \mathbf{m}^{(2,1)} + \mathbf{m}^{(2,2)}, \\ F_{(1,1)}^{(1,2)} &= F_{(1,1)}^{(2,1)} = F_{(1,1)}^{(2,2)} = F_{(2,2)}^{(1,1)} = F_{(2,2)}^{(1,2)} = F_{(2,2)}^{(2,1)} = 0, \\ F_{(1,2)}^{(1,1)} &= F_{(2,1)}^{(2,2)} : \mathbf{m} \rightarrow \mathbf{m}^{(1,1)}, \\ F_{(1,2)}^{(1,2)} &= F_{(2,1)}^{(2,1)} : \mathbf{m} \rightarrow \mathbf{m}^{(1,2)}, \\ F_{(2,1)}^{(1,2)} &= F_{(1,2)}^{(2,1)} : \mathbf{m} \rightarrow \mathbf{m}^{(2,1)}, \\ F_{(2,1)}^{(1,1)} &= F_{(1,2)}^{(2,2)} : \mathbf{m} \rightarrow \mathbf{m}^{(2,2)}. \end{aligned}$$

(ii): Given $\mathbf{m}, \mathbf{p} \in l^1(S_2)$ it is sufficient to observe that

$$\begin{aligned} D(\mathbf{m}) * \mathbf{p} &= D(\mathbf{m}) [\mathbf{p}^{(1,1)} + \mathbf{p}^{(1,2)} + \mathbf{p}^{(2,1)} + \mathbf{p}^{(2,2)}], \\ \mathbf{p} * D(\mathbf{m}) &= [\mathbf{p}^{(1,2)} - \mathbf{p}^{(2,1)}] D(\mathbf{m}). \end{aligned}$$

(iii): An element $\mathbf{m} \in l^1(S_2)$ is singular if and only if $(\mathbf{m}^{(1,2)})^2 = (\mathbf{m}^{(2,1)})^2$ or $\mathbf{m}^{(1,1)} + \mathbf{m}^{(1,2)} + \mathbf{m}^{(2,1)} + \mathbf{m}^{(2,2)} = 0$. Besides $\mathbf{q}_0 \in J(l^1(S_2))$ if and only if

$\delta_{(1,2)} - \mathbf{p} * \mathbf{q}_0$ is regular for any $\mathbf{p} \in l^1(S_2)$ (cf. [3], p. 69). If for a fixed $\mathbf{p}$ we write $\widehat{\mathbf{p}} = \delta_{(1,2)} - \mathbf{p} * \mathbf{q}_0$ then $\widehat{\mathbf{p}} = \delta_{(1,2)} - (\mathbf{p}^{(1,2)} - \mathbf{p}^{(2,1)}) \mathbf{q}_0$. Hence $\widehat{\mathbf{p}}^{(1,2)} = 1$, $\widehat{\mathbf{p}}^{(2,1)} = 0$ and $\widehat{\mathbf{p}}^{(1,1)} + \widehat{\mathbf{p}}^{(1,2)} + \widehat{\mathbf{p}}^{(2,1)} + \widehat{\mathbf{p}}^{(2,2)} = 1$, i.e., $\widehat{\mathbf{p}}$ becomes regular. Thus $\mathbf{q}_0 \in J(l^1(S_2))$ and the claim follows by (8).

(iv): We observe that

$$J(l^1(S_2)) = \{\mathbf{m} \in l^1(S_2) : \mathbf{m}^{(1,1)} + \mathbf{m}^{(2,2)} = \mathbf{m}^{(1,2)} = \mathbf{m}^{(2,1)} = 0\}$$

and by a dimensionality argument the claim follows. $\square$

Example 3. The following is the list of non-zero complex homomorphisms on $l^1(S_3)$ induced by homomorphisms $h : (S_3, \circ) \rightarrow (\{-1, 0, 1\}, \cdot)$:

$$(9) \quad \begin{aligned} h_0(\mathbf{m}) &= \sum_{g \in S_3} \mathbf{m}^g, \\ h_1(\mathbf{m}) &= \mathbf{m}^{(1,2,3)} + \mathbf{m}^{(1,3,2)} + \mathbf{m}^{(2,1,3)} + \mathbf{m}^{(2,3,1)} + \mathbf{m}^{(3,1,2)} + \mathbf{m}^{(3,2,1)}, \\ h_2(\mathbf{m}) &= \mathbf{m}^{(1,2,3)} - \mathbf{m}^{(1,3,2)} + \mathbf{m}^{(2,1,3)} - \mathbf{m}^{(2,3,1)} - \mathbf{m}^{(3,1,2)} - \mathbf{m}^{(3,2,1)}, \\ h_3(\mathbf{m}) &= \mathbf{m}^{(1,2,3)} - \mathbf{m}^{(1,3,2)} - \mathbf{m}^{(2,1,3)} + \mathbf{m}^{(2,3,1)} + \mathbf{m}^{(3,1,2)} - \mathbf{m}^{(3,2,1)}, \\ h_4(\mathbf{m}) &= \mathbf{m}^{(1,2,3)} + \mathbf{m}^{(1,3,2)} - \mathbf{m}^{(2,1,3)} - \mathbf{m}^{(2,3,1)} - \mathbf{m}^{(3,1,2)} + \mathbf{m}^{(3,2,1)}. \end{aligned}$$

If $D \in \mathcal{Z}^1(l^1(S_3))$ the matrix $M = [\langle D(\delta_g), \delta^h \rangle]_{g,h \in \text{Inv}(S_3)}$ has the form

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & c_2 & -c_2 & -c_1 \\ c_3 - c_1 - c_4 & c_5 & -c_2 - c_6 & c_2 - c_7 & c_1 - c_8 \\ c_8 & c_2 - c_3 & c_7 - c_2 & -c_5 & c_4 & c_6 \\ c_8 - c_2 - c_3 & c_2 + c_6 & c_4 & -c_5 & c_7 \\ c_3 & c_1 - c_4 & -c_1 - c_8 & -c_7 & -c_6 & c_5 \end{bmatrix}.$$

By (9) the Jacobson radical of $l^1(S_3)$ is contained in the subspace S of $l^1(S_3)$ defined as

$$S : \mathbf{m}^{(1,2,3)} = \mathbf{m}^{(2,1,3)} = \mathbf{m}^{(1,3,2)} + \mathbf{m}^{(3,2,1)} = \mathbf{m}^{(2,3,1)} + \mathbf{m}^{(3,1,2)} = 0.$$

Therefore, if $\text{Im}(D) \subseteq J(l^1(S_3))$ the following identities must hold

$$c_1 = c_3 = c_4 = c_5 = c_8 = 0 \text{ and } c_2 = -c_6 = c_7.$$

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